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A decomposition technique for integrable functions with applications to the divergence problem

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ABSTRACT

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain that can be written as $\Omega = \bigcup_t \Omega_t$, where $\{\Omega_t\}_{t\in\Gamma}$ is a countable collection of domains with certain properties. In this work, we develop a technique to decompose a function $f \in L^1(\Omega)$, with vanishing mean value, into the sum of a collection of functions $\{f_t - \tilde{f}_t\}_{t\in\Gamma}$ subordinated to $\{\Omega_t\}_{t\in\Gamma}$ such that $\operatorname{supp}(f_t - \tilde{f}_t) \subset \Omega_t$ and $\int f_t - \tilde{f}_t = 0$. As an application, we use this decomposition to prove the existence of a solution in weighted Sobolev spaces of the divergence problem $\operatorname{div} \mathbf{u} = f$ and the well-posedness of the Stokes equations on Hölder-α domains and some other domains with an external cusp arbitrarily narrow. We also consider arbitrary bounded domains. The weights used in each case depend on the type of domain.

1. Introduction

In this paper we show a kind of atomic decomposition for an integral function $f \in L^1(\Omega)$ if Ω is a bounded domain which can be written as the union of a countable collection of domains $\{\Omega_t\}_{t\in\Gamma}$ with certain properties. This result is based on a decomposition developed by Bogovskii in [6], where Γ is finite. The goal of this result is to write a function f with $\int f = 0$ as the sum of a collection of functions $\{f_t - \tilde{f}_t\}_{t\in\Gamma}$ such that $\operatorname{supp}\{f_t - \tilde{f}_t\} \subset \Omega_t$ and $\int_{\Omega_t} f_t - \tilde{f}_t = 0$. As Bogovskii did in his paper we use this decomposition to study the existence of solutions of the divergence problem, and posteriorly the well-posedness of the Stokes equations.

Let us introduce the divergence problem for a bounded domain $\Omega \subset \mathbb{R}^n$. Given $f \in L^p(\Omega)$, with vanishing mean value and $1 < p < \infty$, the divergence problem deals with the existence of a solution $\mathbf{u}$ in the Sobolev space $W_0^{1,p}(\Omega)^n$ of $\operatorname{div} \mathbf{u} = f$ satisfying

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leqslant C_\Omega \|f\|_{L^p(\Omega)}, \tag{1.1}$$

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where $D\mathbf{u}$ is the differential matrix of $\mathbf{u}$. This problem has been widely studied and it has many applications, for example, in the particular case $p = 2$, it is fundamental for the variational analysis of the Stokes equations (see [14]). It is also well known for its relation with some inequalities such as Korn and Sobolev–Poincaré.

Consequently, several methods have been developed to prove the existence of a solution of $\operatorname{div}\mathbf{u} = f$ satisfying (1.1) under different assumptions on the domain (see for example [3,5–7,12,19]).

On the other hand, this result fails if Ω has an external cusp or arbitrarily narrow "corridors", see for example [2] and [13]. However, the existence of solutions of the divergence problem holds in some of these irregular domains if we consider weighted Sobolev spaces with an estimate weaker than (1.1). A similar analysis can be done for its related results. Since the non-existence of standard solutions arises because of the bad behavior of the boundary, it seems natural to work with weights involving the distance to the boundary of Ω or a subset of it. The following are some papers considering the divergence problem or related results in weighted Sobolev spaces [1,4,7,9,10,20].

Another point of interest is the characterization of the domains where there exists a standard solution of the divergence equation. This problem has been completely solved if Ω is a bounded planar simply connected domain where it was proved that there exists a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^2$ of $\operatorname{div}\mathbf{u} = f$ satisfying (1.1) if and only if Ω is a John domain. The case $1 < p < 2$ was published in [3] while $2 \leqslant p < \infty$ was recently shown in [15].

As we mentioned before there are many different approaches to this problem. In the present paper, as it was done in [7] and [11], we use a decomposition of the function f in $\operatorname{div}\mathbf{u} = f$ to generalize results valid on simple domains, such as rectangles or star-shaped domains, to more general cases.

The paper is organized in the following way: In Section 2, we include some notations and preliminary results. In Section 3, we show the main result of this paper, a decomposition technique for integrable functions defined over a bounded domain Ω which is written as the union of a collection of subdomains $\{\Omega_t\}_{t\in\Gamma}$ with some properties. The set Γ is required to have a certain partial order structure. In the following sections we include three different applications of the decomposition developed in Section 3. These sections can be independently read. In Section 4, we show the existence of a weighted right inverse of the divergence operator on arbitrary bounded domains. In Sections 5 and 6, we prove the existence of a solution of the divergence problem and the well-posedness of the Stokes equations on some domains with an external cusp arbitrarily narrow and on bounded Hölder-α domains in $\mathbb{R}^n$. The weights in these two final sections are more specific than the one used in Section 4. More precisely, the weights are related to the distance to the cusp and to the distance to the boundary of the domain respectively.

2. Preliminaries and notations

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given a measurable positive function $\omega : \Omega \to R_{>0}$ we denote with $L^p(\Omega,\omega)$ the weighted space with norm

$$\|f\|_{L^p(\Omega,\omega)} = \|f\omega\|_{L^p(\Omega)},$$

and with $W_0^{1,p}(\Omega,\omega)$ the weighted Sobolev space defined as the closure of $C_0^\infty(\Omega)$ with norm

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \|Du\|_{L^p(\Omega,\omega)},$$

where Du is the differential matrix of u. Observe that the seminorm $\|Du\|$ is a norm in the trace zero space.

We say that Ω satisfies $(\operatorname{div})_p$, for $1 < p < \infty$, with constant C_Ω if for any $f \in L_0^p(\Omega) := \{g \in L^p(\Omega)$: g has vanishing mean value$\}$ there is a solution $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ of $\operatorname{div}\mathbf{u} = f$ satisfying (1.1). We also use C_A to denote a constant depending on A, where A is not necessarily a domain.

In the next lemma we compare C_Ω with $C_{\hat\Omega}$, where Ω is a domain obtained by applying an affine function to a domain $\hat\Omega$ satisfying $(\operatorname{div})_p$. This result is standard and the proof uses the Piola transform.