J. Math. Anal. Appl. 418 (2014) 108–120

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Topologized Hilbert spaces ☆

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ARTICLE INFO

Article history:
Received 19 June 2013
Available online 1 April 2014
Submitted by D. Blecher

Keywords:
Hilbert spaces
Densely defined operators
*-Algebras
Banach *-algebras

ABSTRACT

Let $\mathcal{H}_{\mathcal{P}}$ be a Hausdorff topological vector space with the underlying vector space $\mathcal{H}$ being a Hilbert space such that $\mathcal{P}$ is coarser than the norm topology. A densely defined $\mathcal{P}$-$\mathcal{P}$-continuous operator on $\mathcal{H}$ is called $\mathcal{P}$-maximal if it has no non-trivial $\mathcal{P}$-$\mathcal{P}$-continuous extension, and it is said to be $\mathcal{P}$-adjointable if its adjoint is also $\mathcal{P}$-$\mathcal{P}$-continuous.

We show that if $\mathcal{P}$ is locally convex, the collection $\mathfrak{M}^{\star}_{\mathcal{P}}(\mathcal{H})$ of all densely defined $\mathcal{P}$-maximal $\mathcal{P}$-adjointable operators is a *-algebra under the multiplication given by the $\mathcal{P}$-maximal extension of the composition and the involution $^{\diamond}$ given by the $\mathcal{P}$-maximal extension of the adjoint. Examples include rigged Hilbert spaces and O^*-algebras.

In the general (not necessarily locally convex) case, we associate with $\mathcal{H}_{\mathcal{P}}$ a *-algebra $\mathfrak{L}^{\star}_{\mathfrak{b}}(\widetilde{\mathcal{H}_{\mathcal{P}}})$ which is a *-subalgebra of $\mathfrak{M}^{\star}_{\mathcal{P}}(\mathcal{H})$ when $\mathcal{P}$ is locally convex. If $\mathcal{P}$ is the measure topology on $\mathcal{H}$ corresponding to a tracial von Neumann algebra $\mathcal{M} \subseteq \mathfrak{L}(\mathcal{H})$, then the image of the representation of the measurable operator algebra on the completion $\widetilde{\mathcal{H}_{\mathcal{P}}}$ of $\mathcal{H}$ with respect to $\mathcal{P}$, can be regarded as a *-subalgebra of $\mathfrak{L}^{\star}_{\mathfrak{b}}(\widetilde{\mathcal{H}_{\mathcal{P}}})$. In the case when $\mathcal{P}$ is normable, it is shown that $\mathfrak{L}^{\star}_{\mathfrak{b}}(\widetilde{\mathcal{H}_{\mathcal{P}}})$ is a Banach *-algebra. Examples of such Banach *-algebras include $\mathfrak{L}^{\star}_{L_\infty[0,1]}(L_2[0,1]) := \{\Psi \in \mathfrak{B}(L_2[0,1]) \colon \Psi(L_\infty[0,1]) \subseteq L_\infty[0,1]; \Psi^*(L_\infty[0,1]) \subseteq L_\infty[0,1]\}$ (under a suitable norm) as well as $\mathfrak{L}^{\star}_{\mathfrak{T}(\ell^2)}(\mathfrak{S}(\ell^2)) := \{\Phi \in \mathfrak{B}(\mathfrak{S}(\ell^2)) \colon \Phi(\mathfrak{T}(\ell^2)) \subseteq \mathfrak{T}(\ell^2); \Phi^*(\mathfrak{T}(\ell^2)) \subseteq \mathfrak{T}(\ell^2)\}$, where $\mathfrak{S}(\ell^2)$ and $\mathfrak{T}(\ell^2)$ are the spaces of Hilbert–Schmidt operators and of trace-class operators respectively, on ℓ^2.

1. Introduction

It is an interesting question of when a collection of densely defined operators on a Hilbert space forms a *-algebra. In the literature, there is the notion of O^*-algebra (see e.g. [2] or [9]) that roughly speaking concerns operators with their domains (as well as those of their adjoints) containing a fixed dense subspace of $\mathcal{H}$.

Motivated by the construction in [5], we consider another approach here. We equip a Hilbert space $\mathcal{H}$ with a Hausdorff vector topology $\mathcal{P}$ that is coarser than the original topology. We say that a densely defined $\mathcal{P}$-continuous operator $S\colon \operatorname{dom} S \to \mathcal{H}$ is "$\mathcal{P}$-maximal" if S is the only $\mathcal{P}$-continuous operator extending S.

☆ This work is supported by the National Natural Science Foundation of China (11071126).
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http://dx.doi.org/10.1016/j.jmaa.2014.03.073

Moreover, S is said to be "$\mathcal{P}$-adjointable" if its adjoint S^* is also $\mathcal{P}$-continuous. If $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$ is the collection of all $\mathcal{P}$-maximal $\mathcal{P}$-adjointable operators, then any densely defined $\mathcal{P}$-adjointable $\mathcal{P}$-continuous operator can be extended uniquely to an element in $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$.

Suppose that $\mathcal{P}$ is locally convex. For any densely defined operators S and T, we set $S+T$ to be the sum with domain $\operatorname{dom} S \cap \operatorname{dom} T$ and $S \circ T$ to be the composition with domain $T^{-1}(\operatorname{dom} S)$. If $S, T \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$, we will show that both $S+T$ and $S \circ T$ are densely defined and their $\mathcal{P}$-maximal extensions lie in $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$. Furthermore, the $\mathcal{P}$-maximal extension, $S^\diamond$, of the adjoint S^* (which will be densely defined in this case) is also in $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$. These give a *-algebra structure on $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$ (see Theorem 2.6). For any $S \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$, the relation $S^\diamond = S$ is the same as $S^* \subseteq S$. If $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})_h = \{S \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H}):\ S^\diamond = S\}$, there is a canonical generating cone $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})_+$ of $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})_h$ containing the identity map $I_\mathcal{H}$ such that $RSR^\diamond \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})_+$ for every $S \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})_+$ as well as $R \in \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$.

In Section 3, we will give several examples of such *-algebras. First of all, the concrete example of ℓ^2 equipped with the topology $\mathcal{P}$ induced by the ℓ^∞-norm will be investigated (in Example 3.6). Moreover, if $K \subseteq \mathcal{H}$ is any norm-dense subspace, one can define a topology $\mathcal{P}_K$ on $\mathcal{H}$ such that $\mathfrak{M}^\star_{\mathcal{P}_K}(\mathcal{H})$ is the largest O^*-algebra on K (see e.g. [9, Proposition 2.1.8]). Furthermore, if $\mathcal{A}$ is an O^*-algebra on K, and $\mathfrak{t}_\mathcal{A}$ is the corresponding graph topology, there exists a topology $\mathcal{P}$ on $\mathcal{H}$ that is coarser than the norm topology and is compatible with $\mathfrak{t}_\mathcal{A}$ such that $\mathcal{A}$ can be regarded as a *-subalgebra of $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$.

In the case of general vector topology $\mathcal{P}$ (i.e. not necessarily locally convex), one can also associate with it a *-algebra $\mathfrak{L}^\star_\mathbf{s}(\widetilde{\mathcal{H}_\mathcal{P}})$ by considering the closure of the *-algebra $\mathfrak{B}^\star_\mathcal{P}(\mathcal{H})$ (see (1.1)) in some semi-topological *-algebra. If $\mathcal{P}$ happens to be locally convex, then $\mathfrak{L}^\star_\mathbf{s}(\widetilde{\mathcal{H}_\mathcal{P}})$ can be regarded a *-subalgebra of $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$ (Theorem 4.1). In the case when $\mathcal{P}$ is the measure topology associated with a tracial von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$, the completion of $\mathcal{M}$ under the measure topology can be regarded as a *-subalgebra of $\mathfrak{L}^\star_\mathbf{s}(\widetilde{\mathcal{H}_\mathcal{P}})$.

We will also look at the situation when $\mathcal{P}$ is normable in Section 5. In this case, a certain *-subalgebra $\mathfrak{L}^\star_\mathbf{b}(\widetilde{\mathcal{H}_\mathcal{P}})$ of $\mathfrak{L}^\star_\mathbf{s}(\widetilde{\mathcal{H}_\mathcal{P}})$ is actually a Banach *-algebra. If it happens that the norm topology on $\mathcal{H}$ is an interpolation between the completion of $\mathcal{H}_\mathcal{P}$ and the dual norm on the dual space of $\mathcal{H}_\mathcal{P}$ (which can be regarded as a subspace of $\mathcal{H}$), then $\mathfrak{B}^\star_\mathcal{P}(\mathcal{H}) = \mathfrak{L}^\star_\mathbf{b}(\widetilde{\mathcal{H}_\mathcal{P}}) = \mathfrak{L}^\star_\mathbf{s}(\widetilde{\mathcal{H}_\mathcal{P}}) = \mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$ (Proposition 5.1). In particular, this situation happens when $\mathcal{H}$ is the GNS construction of a faithful normal state ϕ of a von Neumann algebra $\mathcal{M}$ and $\mathcal{P}$ is the topology defined by the natural embedding of $\mathcal{H}$ into the predual $\mathcal{M}_*$ through ϕ (Corollary 5.3). Examples also include $\mathfrak{L}^\star_{\mathfrak{T}(\ell^2)}(\mathfrak{S}(\ell^2))$ and $\mathfrak{L}^\star_{L_\infty[0,1]}(L_2[0,1])$ as in the abstract. In [4], we will show that any derivation on $\mathfrak{M}^\star_\mathcal{P}(\mathcal{H})$ is automatically continuous and is inner, when $\mathcal{P}$ is normable.

Notation 1.1. All vector spaces are over the complex field. We denote by $L(X)$ the set of all linear maps from a vector space X to itself. If $X_\mathcal{P}$ is a Hausdorff topological vector space, we denote by $(X_\mathcal{P})'$ (or $X'_\mathcal{P}$) and $\mathfrak{L}(X_\mathcal{P})$ the set of all $\mathcal{P}$-continuous linear functionals on X and the set of all $\mathcal{P}$-$\mathcal{P}$-continuous linear maps from X to X, respectively. We also denote by $\widetilde{X_\mathcal{P}}$ the completion of $X_\mathcal{P}$, equipped with the topology induced by $\mathcal{P}$ (again denoted by $\mathcal{P}$). If $\mathcal{P}$ is defined by a norm $\|\cdot\|$, we will also write $X_{\|\cdot\|}$ for $X_\mathcal{P}$. If $\mathcal{H}$ is a Hilbert space, we use $\mathfrak{B}(\mathcal{H})$ to denote the collection of bounded linear maps from $\mathcal{H}$ to $\mathcal{H}$. Furthermore, if $R: \operatorname{dom} R \to \mathcal{H}$ is a densely defined operator on $\mathcal{H}$, we denote by R^* the adjoint of R.

Definition 1.2. $(\mathcal{H}, \mathcal{P})$ is called a *topologized Hilbert space* if $\mathcal{H}_\mathcal{P}$ is a Hausdorff topological vector space with $\mathcal{H}$ being a Hilbert space such that $\mathcal{P}$ is coarser than the topology induced by the inner product. In this case, we set

$$\mathfrak{B}^\star_\mathcal{P}(\mathcal{H}) := \{R \in \mathfrak{B}(\mathcal{H}):\ R \text{ and } R^* \text{ are } \mathcal{P}\text{-continuous}\}. \tag{1.1}$$

A topologized Hilbert space $(\mathcal{H}, \mathcal{P})$ is said to be *metrizable* (respectively, *locally convex*) if $\mathcal{P}$ is metrizable (respectively, locally convex). On the other hand, if $\mathcal{H}'_\mathcal{P}$ separates points of $\mathcal{H}$, we say that $(\mathcal{H}, \mathcal{P})$ is *separated.*

It is clear that if $(\mathcal{H}, \mathcal{P})$ is locally convex, then it is separated.