J. Math. Anal. Appl. 418 (2014) 121–141

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A model space approach to some classical inequalities for rational functions

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ARTICLE INFO

Article history:
Received 7 October 2013
Available online 26 March 2014
Submitted by Richard M. Aron

Keywords:
Rational function
Hardy space
Bernstein inequality
Nikolskii inequality

ABSTRACT

We consider the set $\mathcal{R}_n$ of rational functions of degree at most $n \geqslant 1$ with no poles on the unit circle $\mathbb{T}$ and its subclass $\mathcal{R}_{n,r}$ consisting of rational functions without poles in the annulus $\{\xi\colon r \leqslant |\xi| \leqslant \frac{1}{r}\}$. We discuss an approach based on the model space theory which brings some integral representations for functions in $\mathcal{R}_n$ and their derivatives. Using this approach we obtain L^p-analogs of several classical inequalities for rational functions including the inequalities by P. Borwein and T. Erdélyi, the Spijker Lemma and S.M. Nikolskii's inequalities. These inequalities are shown to be asymptotically sharp as n tends to infinity and the poles of the rational functions approach the unit circle $\mathbb{T}$.

1. Introduction

The goal of this paper is to give a unified approach to classical inequalities for rational functions. This approach is based on integral representations for rational functions and their derivatives. It makes possible to recover several known results and obtain their L^p-analogs where the estimate is given not only in terms of the degree of a rational function but also in terms of the distance from the poles to the boundary.

1.1. Notations

Let $\mathcal{P}_n$ be the space of complex analytic polynomials of degree at most $n \geqslant 1$ and let

$$\mathcal{R}_n = \left\{ \frac{P}{Q}\colon\ P, Q \in \mathcal{P}_n,\ Q(\xi) \neq 0 \text{ for } |\xi| = 1 \right\}$$

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[1] The author is supported by the Chebyshev Laboratory (St. Petersburg State University) under RF Government Grant 11.G34.31.0026, by JSC "Gazprom Neft", by RFBR Grant 12-01-00434 and by Dmitry Zimin's Dynasty Foundation.

http://dx.doi.org/10.1016/j.jmaa.2014.03.059
0022-247X/

be the set of rational functions of degree at most n (where $\deg \frac{P}{Q} = \max(\deg P, \deg Q)$) without poles on the unit circle $\mathbb{T} = \{\xi \in \mathbb{C}\colon |\xi| = 1\}$. We denote by $\|f\|_{L^p}$, $0 < p \leqslant \infty$, the standard norms (quasinorms if $p \in (0,1)$) of the spaces $L^p(\mathbb{T}, m)$, where m stands for the normalized Lebesgue measure on $\mathbb{T}$. Denote by $\mathbb{D} = \{\xi \in \mathbb{C}\colon |\xi| < 1\}$ the unit disc of the complex plane and by $\overline{\mathbb{D}} = \{\xi \in \mathbb{C}\colon |\xi| \leqslant 1\}$ its closure. For a given $r \in (0,1)$, we finally introduce the subset

$$\mathcal{R}_{n,r} = \left\{\frac{P}{Q}\colon\ P, Q \in \mathcal{P}_n,\ Q(\xi) \neq 0 \text{ for } r \leqslant |\xi| \leqslant \frac{1}{r}\right\}$$

of $\mathcal{R}_n$, consisting of rational functions of degree at most n without poles in the annulus $\{\xi\colon\ r \leqslant |\xi| \leqslant \frac{1}{r}\}$.

We also introduce some notations specific to the theory of model subspaces of the Hardy space H^p, $1 \leqslant p \leqslant \infty$. Denote by $\mathrm{Hol}(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. The Hardy space $H^p = H^p(\mathbb{D})$, $1 \leqslant p < \infty$, is defined as follows:

$$H^p = \left\{f \in \mathrm{Hol}(\mathbb{D})\colon\ \|f\|^p_{H^p} = \sup_{0 \leqslant \rho < 1} \int_{\mathbb{T}} \left|f(\rho\xi)\right|^p \mathrm{d}m(\xi) < \infty\right\}.$$

As usual, we denote by H^∞ the space of all bounded analytic functions in $\mathbb{D}$. We denote by $\langle \cdot,\cdot \rangle$ the usual scalar product on $L^2(\mathbb{T}, m)$ or H^2,

$$\langle f, g \rangle = \int_{\mathbb{T}} f(\xi)\overline{g(\xi)}\,\mathrm{d}m(\xi). \tag{1.1}$$

For any $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^{n} b_{\lambda_k}, \qquad b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z},$$

b_λ being the elementary Blaschke factor associated with $\lambda \in \mathbb{D}$. Define the model subspace K_{B_σ} of the Hardy space H^2 by

$$K_{B_\sigma} = \left(B_\sigma H^2\right)^\perp = H^2 \ominus B_\sigma H^2.$$

The subspace K_{B_σ} consists of rational functions of the form P/Q, where $P \in \mathcal{P}_{n-1}$ and Q is a polynomial of degree n with the zeros $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ of corresponding multiplicities.

For any Blaschke product B and $\xi \in \mathbb{D}$, the function

$$k^B_\xi(z) = \frac{1 - \overline{B(\xi)}B(z)}{1 - \bar{\xi} z}$$

is the reproducing kernel of the model space K_B corresponding to ξ, that is, $f(\xi) = \langle f, k^B_\xi \rangle$, $f \in K_B$ (see, e.g., [20, p. 199]). Indeed, for any $f \in K_B$ and $\xi \in \mathbb{D}$,

$$f(\xi) = \left\langle f, \frac{1}{1 - \bar{\xi} z} \right\rangle = \left\langle f, k^B_\xi \right\rangle. \tag{1.2}$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that $(1 - \bar{\xi} z)^{-1} - k^B_\xi(z) \in BH^2$ and $f \perp BH^2$. In the case when B is a finite Blaschke product we have $K_B \subset C(\overline{\mathbb{D}})$ (i.e., any $f \in K_B$ is continuous in the closed disc) and $k^B_\xi(z) \in C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ as a function of ξ and z. Thus, the formula $f(\xi) = \langle f, k^B_\xi \rangle$ remains true for $\xi \in \mathbb{T}$.