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Blow-up analysis for solutions to Neumann boundary value problem

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ABSTRACT

In this paper, we will analyze the blow-up behavior of solution sequences satisfying a conformal invariant equation defined on a compact 2-dimensional surface (M, g) with boundary. We will provide some accurate point-wise estimates for the profile of these sequences.

1. Introduction

Let (M, g) be a compact 2-dimensional surface with smooth boundary ∂M. In this paper we want to study the following geometric equations:

$$\begin{cases} -\Delta_g u = 2e^{2u} - K_g, & \text{in } M^o, \\ \dfrac{\partial u}{\partial n} = ce^u - h_g, & \text{on } \partial M. \end{cases} \tag{1.1}$$

Here Δ_g stands for the Laplace Beltrami operator with respect to g, K_g is the Guassian curvature of M, h_g is geodesic curvature of ∂M, c is a given constant. Problem (1.1) is also called Liouville equation with Neumann boundary value condition. The geometric meaning of problem (1.1) is that u is a solution if and only if, up to some harmless positive constant, the Gaussian curvature of M with the metric $\tilde{g} = e^{2u} g$ is 2 and the geodesic curvature of ∂M is c. Moreover, if we define the functional

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$$E_B(u) = \int_M \left\{ \frac{1}{2}|\nabla u|^2 + K_g u - e^{2u} \right\} dv + \int_{\partial M} h_g u - ce^u \, d\sigma, \tag{1.2}$$

then problem (1.1) is the Euler–Lagrange system for $E_B(u)$. The key property of the functional is of course its conformal invariance. Therefore, the boundary conditions should also be conformally invariant. Conformal invariance on the one hand makes the solution space non-compact, but on the other hand allows for a control of limits of solutions via a blow-up analysis. The main point of the paper then is an analytical investigation of these blow-up sequences. In particular, we shall show the regularity of solutions and identify the blow-up behavior for limits of sequences of solutions. In other words, we analytically understand the non-compactness of the solution space.

When M is closed Riemann surface, i.e. $\partial M = \emptyset$, the Liouville functional is

$$E(u) = \int_M \left\{ \frac{1}{2}|\nabla u|^2 + K_g u - e^{2u} \right\} dv,$$

and the Euler–Lagrange equation is

$$-\Delta_g u = 2e^{2u} - K_g, \quad \text{in } M,$$

which is called as Liouville equation. Liouville [11] studied this equation in the plane, that is, for $K_g = 0$. The Liouville equation is a basic equation for the complex analysis and differential geometry for 2-dimensional surfaces, in particular it shows up in the prescribing curvature problem. Liouville equation has a lot of variants, named as Liouville-type equation. The typical equation of these variants is mean field equation on M, i.e.

$$-\Delta_g w = \lambda \left(\frac{Ve^w}{\int_M Ve^w \, dv_g} - W \right), \quad \text{on } M, \tag{1.3}$$

where V is a nonnegative function on M, W be a function with $\int_M W \, dv_g = 1$, $\lambda \in \mathbb{R}$. One interesting subject is to solve the mean field equation. When λ is big enough, the solution sequences will blow up. Many works have been taken to understand the blow-up analysis of solution sequences of the Liouville-type problem on manifold without boundary in the past years. A full blow-up theory for the Liouville-type equation was established in [11,2,9,8,1] and for a Toda system in [4,3] and the reference therein. In those references, it was established that the key analytical points are that singularities in solutions u_n of the equations on closed surfaces, or, more generally with bounded energy $\int e^{2u_n}$, can form only at isolated points x, with a quantized loss of "energy", where the limit $u_n(x)$ tends to infinity. Away from those singularities, u_n remains either uniformly bounded or converges to $-\infty$ which, in fact, is a regular situation for the field ϕ with $u = \log \phi$. At those isolated singularities, rescaling produces an entire solution of the Liouville equation of finite energy $\int_{\mathbb{R}^2} e^{2u}$ in the plane which then can be compactified to a solution on the 2-sphere. Moreover, it was also established the energy identity for solutions, the blow-up values at the blow-up points, and the profile of solutions near the blow-up point.

In this paper, we will investigate the blow-up behavior for Neumann boundary problem (1.1), particularly the blow-up value and the profile of solutions near the blow-up point.

Theorem 1.1. *Assume that u_n satisfies:*

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - K_g, & \text{in } M^o, \\ \dfrac{\partial u_n}{\partial n} = ce^{u_n} - h_g, & \text{on } \partial M, \end{cases} \tag{1.4}$$