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On the accuracy of the approximation of the complex exponent by the first terms of its Taylor expansion with applications ☆

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ABSTRACT

A modification of the Taylor expansion for the complex exponential function e^{ix}, $x \in \mathbf{R}$, is proposed yielding precise moment-type estimates of the accuracy of the approximation of a Fourier transform by the first terms of its Taylor expansion. Moreover, a precise upper bound for the third moment of a probability distribution in terms of the absolute third moment is established. Based on these results, new precise bounds for Fourier–Stieltjes transforms of probability distribution functions and for their derivatives are obtained that are uniform in classes of distributions with prescribed first three moments.

1. Introduction and notations

Significance of the complex exponential function e^{ix}, $x \in \mathbf{R}$, can hardly be overestimated. In particular, complex exponential function forms a kernel in the so-called *Fourier transform* which is one of the most popular integral transforms used in mathematics and its various applications. That is why it is very important to study the complex exponential function thoroughly. Actually, a lot of its properties and representations are already known. However, while treating another problems connected with applications of Fourier transforms in probability theory and statistics, the author came across with one of such a representation that seemed to be unknown before and simultaneously was critical for the problems under consideration. This representation is a modification of the famous Taylor formula, which is stated in Section 2. Section 3 gives an application of the obtained analytical results to Fourier transform of probability measures (characteristic functions). In this section a precise moment inequality for a probability distribution is proved yielding

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precise moment-type bounds for the corresponding characteristic function and its derivatives. The obtained bounds are useful, for example, in such fields of probability theory and statistics as limit theorems and stability of characterizations. All the proofs are accumulated in Section 4.

As is well known, the remainder term

$$r_n(x) = e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!}, \quad x \in \mathbf{R},\ n \in \mathbf{N},$$

in the Taylor expansion of the complex exponent admits the representation

$$r_n(x) = \frac{(ix)^n}{n!} + o\big(x^n\big), \quad x \to 0.$$

However, this representation can hardly be used in practice because of the presence of the term $o(x^n)$, and usually, one uses the bound

$$\big|r_n(x)\big| \leqslant \frac{|x|^n}{n!}, \quad x \in \mathbf{R},\ n \in \mathbf{N}, \tag{1}$$

which is precise in the sense that the factor $1/n!$ on the r.-h. side cannot be made less (since $\lim_{x\to 0} |r_n(x)|/|x|^n = 1/n!$). However, in 1991 H. Prawitz [25] managed to improve precise bound (1) by rearranging a part of the remainder (however, always a smaller part) to the main term. He proved that

$$\left|r_n(x) - \frac{n}{2(n+1)} \cdot \frac{(ix)^n}{n!}\right| \leqslant \frac{n+2}{2(n+1)} \cdot \frac{|x|^n}{n!}, \quad x \in \mathbf{R},\ n \in \mathbf{N}, \tag{2}$$

whence (1) immediately follows by virtue of the evident inequalities

$$\big|r_n(x)\big| \leqslant \left|r_n(x) - \frac{n}{2(n+1)} \cdot \frac{(ix)^n}{n!}\right| + \frac{n}{2(n+1)} \cdot \frac{|x|^n}{n!} \leqslant \frac{|x|^n}{n!}, \quad x \in \mathbf{R},\ n \in \mathbf{N}.$$

The advantage of bound (2) as compared with (1) is especially noticeable, if x becomes an integration variable. Some examples are given in Section 3 below.

Inequality (2) stipulates natural questions: if a larger part of the remainder is rearranged to the main term, will the factor $(n+2)/(2(n+1)!)$ on the r.-h. side of (2) become less or not? If yes, then what is its least possible value and will the sum of the coefficients at the corresponding main term and remainder still be equal to one or will it increase? To answer these questions, we propose to consider for each $n \in \mathbf{N}$ a family of the bounds of the form

$$\left|r_n(x) - \lambda \cdot \frac{(ix)^n}{n!}\right| \leqslant q_n(\lambda) \cdot \frac{|x|^n}{n!}, \quad 0 \leqslant \lambda \leqslant 1/2,$$

which should hold for all $x \in \mathbf{R}$ with some $q_n(\lambda) > 0$. It is easy to see that bound (2) is a particular case of the proposed inequality with

$$\lambda = \frac{n}{2(n+1)}, \qquad q_n(\lambda) = \frac{n+2}{2(n+1)}.$$

Of course, we are interested in the least possible values of the coefficients $q_n(\lambda)$ providing the above inequalities.