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Positive solutions to integral systems with weight and Bessel potentials

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ABSTRACT

In this paper, we consider the integral system with weight and the Bessel potentials:

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y)u(y)^p v(y)^q}{|y|^\sigma}\,dy, \\ v(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y)v(y)^p u(y)^q}{|y|^\sigma}\,dy, \end{cases}$$

where $u, v > 0$, $\sigma \geqslant 0$, $0 < \alpha < n$, $p + q = \gamma \geqslant 2$ and $g_\alpha(x)$ is the Bessel potential of order α. First, we get the integrability by regularity lifting lemma. Then we also establish the regularity of the positive solutions. Afterwards, by the method of moving planes in integral forms, we show that the positive solutions are radially symmetric and monotone decreasing about the origin. Finally, by an extension of the idea of Lei [14] and analytical techniques, we get the decay rates of solutions when $|x| \to \infty$.

1. Introduction

In this paper, we consider the following integral system with weight and the Bessel potentials:

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y)u(y)^p v(y)^q}{|y|^\sigma}\,dy, \\ v(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y)v(y)^p u(y)^q}{|y|^\sigma}\,dy, \end{cases} \tag{1.1}$$

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where $u, v > 0$, $\sigma \geqslant 0$, $0 < \alpha < n$, $p + q = \gamma \geqslant 2$ and $g_\alpha(x)$ is the Bessel potential of order α. Here

$$g_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{(-\frac{\pi}{t}|x|^2 - \frac{t}{4\pi})} t^{(\alpha-n)/2} \frac{dt}{t}.$$

Integral system (1.1) is associated with the following partial differential equations (PDEs)

$$\begin{cases} (I-\Delta)^{\alpha/2} u = \dfrac{u^p v^q}{|y|^\sigma}, & u > 0, \\ (I-\Delta)^{\alpha/2} v = \dfrac{v^p u^q}{|y|^\sigma}, & v > 0. \end{cases} \tag{1.2}$$

When $\sigma = 0$, (1.2) becomes the following PDEs (cf. [23])

$$\begin{cases} (I-\Delta)^{\alpha/2} u = u^p v^q, & u > 0, \\ (I-\Delta)^{\alpha/2} v = v^p u^q, & v > 0. \end{cases} \tag{1.3}$$

When $\alpha = 2$, PDEs (1.3) is associated with the nonlinear Klein–Gordon equations and the quintic Schrödinger system (see [1,13,19]).

Lei [14] studied the uniqueness of the positive solution of (1.3) under some assumptions. In addition, he proved the integrability and radial symmetry of positive solutions of integral system. By an iteration he also obtained the estimate of the exponential decay of those solutions near infinity.

Another integral system with weight and the Bessel potential is the following

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y) v(y)^q}{|y|^\beta}\, dy, \\ v(x) = \displaystyle\int_{R^n} \frac{g_\alpha(x-y) u(y)^p}{|y|^\beta}\, dy, \end{cases} \tag{1.4}$$

where $0 \leqslant \beta < \alpha < n$, $1 < p, q < \frac{n-\beta}{\beta}$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\alpha+\beta}{n}.$$

Chen and Yang [2] proved regularity and symmetry of this integral system and obtained that system was actually equivalent to indefinite fractional elliptic system

$$\begin{cases} (-\Delta+I)^{\alpha/2} u = \dfrac{v^q}{|y|^\beta}, & u > 0, \\ (-\Delta+I)^{\alpha/2} v = \dfrac{u^p}{|y|^\beta}, & v > 0. \end{cases} \tag{1.5}$$

If $\alpha = 2$ and $\beta = 0$, (1.5) is the Hamiltonian type system [6]. In the special case, when $p = q$ and $u = v$, system (1.5) becomes

$$(-\Delta+I)^{\frac{\alpha}{2}} u = \frac{u^p}{|y|^\beta}. \tag{1.6}$$

It was known from [17] and [18] that the dynamical behavior of bosons spin-0 particles in relativistic fields can be described by the Schrödinger–Klein–Gordon equation