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Extended spectrum, extended eigenspaces and normal operators

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ABSTRACT

We say that a complex number λ is an extended eigenvalue of a bounded linear operator T on a Hilbert space $\mathcal{H}$ if there exists a nonzero bounded linear operator X acting on $\mathcal{H}$, called extended eigenvector associated to λ, and satisfying the equation $TX = \lambda XT$. In this paper we describe the sets of extended eigenvalues and extended eigenvectors for the product of a positive and a self-adjoint operator which are both injective. We also treat the case of normal operators.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{L}(\mathcal{H})$, then a complex number λ is an extended eigenvalue of T if there is a nonzero operator X such that $TX = \lambda XT$. We denote by the symbol $\sigma_{ext}(T)$ the set of extended eigenvalues of T. The set of all extended eigenvectors corresponding to λ will be denoted by $E_{ext}(\lambda)$.

The concepts of extended eigenvalue and extended eigenvector are closely related to generalization of famous Lomonosov's theorem on existence of non-trivial hyperinvariant subspace for the compact operators on a Banach space, which was done by S. Brown in [4], and Kim, Moore and Pearcy in [7], and is stated as follows:

If an operator T on a Banach space has a non-zero compact eigenvector, then T has a nontrivial hyperinvariant subspace.

The special case, where T commutes with a non-zero compact operator, is the original theorem of Lomonosov [9].

Extended eigenvalues and their corresponding extended eigenvectors were studied by several authors (see for example [1,3,6,8]).

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In [3], Biswas, Lambert and Petrovic have introduced this notion and they have shown that $\sigma_{ext}(V) =]0, +\infty[$ where V is the well-known integral Volterra operator on the space $L^2[0,1]$.

Recently, in [13], Shkarin has shown that there is a compact quasinilpotent operator T for which $\sigma_{ext}(T) = \{1\}$, that which allows to classify this type of operators. In [6], Karaev gave a complete description of the set of extended eigenvectors of V. Moreover, in [10] and [11], Malamud had given the set of extended eigenvectors of some generalized Volterra operators (see [5] for more information about this class of operators).

In this paper we treat a large class of operators, that is the normal operators.

Section 2 introduces the sets of extended eigenvalues and extended eigenvectors for the product of a positive operator by a self-adjoint operator which are both injective. As a consequence, we obtain a characterization of the last sets for an injective self-adjoint operator in terms of its spectral measure. We also give two applications of important classes of injective self-adjoint operators.

Section 3 is dedicated to the normal operators. We generalize our results on the self-adjoint operators to the normal operators case.

Let $T \in \mathcal{L}(\mathcal{H})$, and let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_c(T)$ denote the spectrum, the point and the continuous spectrum of T respectively. Using a theorem of Rosenblum [12], it was established in [2] that

$$\sigma_{ext}(T) \subset \{\lambda \in \mathbb{C}\colon\ \sigma(T) \cap \sigma(\lambda T) \neq \emptyset\}. \tag{1.1}$$

It is known that for any self-adjoint operator $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) \subset \mathbb{R}$ and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$. Obviously, if T is a non-injective self-adjoint operator, then $\sigma_{ext}(T) = \mathbb{C}$. Indeed, for all $\lambda \in \mathbb{C}$, one can take X being a nonzero operator from the kernel of T to itself. In addition, if $\sigma(T) \subset \mathbb{R}^*$ then by (1.1) $\sigma_{ext}(T) \subset \mathbb{R}$. Indeed, $\sigma(\lambda T) = \{\lambda t\colon\ t \in \sigma(T)\}$. Consequently, $\sigma(T) \cap \sigma(\lambda T) = \emptyset$ for all $\lambda \in \mathbb{C} \backslash \mathbb{R}^*$.

So, we are interested in the case when T is an injective non-invertible self-adjoint operator, i.e., $0 \in \sigma_c(T)$.

2. Product of self-adjoint operators

In this section, we characterize the sets of extended eigenvalues and extended eigenvectors of the product of a positive and a self-adjoint operator which are both injective. First we will show some auxiliary results.

Lemma 2.1. *Let $R \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, and let $a > \|R\|$. Then for any $p \in \mathbb{C}[X]$ we have,*

$$\langle p(R)x, y\rangle = p(a)\langle x, y\rangle - \int_{-a}^{a} p'(t)\langle E(]-\infty, t])x, y\rangle\, dt, \quad \forall x, y \in \mathcal{H},$$

where $\langle\,,\rangle$ denotes the standard inner product in $\mathcal{H}$, and E denotes the spectral measure associated to R. In particular, if R is a positive operator, then

$$\langle p(R)x, y\rangle = p(a)\langle x, y\rangle - \int_{0}^{a} p'(t)\langle E([0, t])x, y\rangle\, dt, \quad \forall x, y \in \mathcal{H}.$$

Proof. First, recall that the indicator function of a subset $\Omega \subset \mathbb{R}$ is defined by

$$\mathbf{1}_\Omega(t) = \begin{cases} 1 & \text{if } t \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\int_{-a}^{a} p'(t)\langle E(]-\infty, t])x, y\rangle\, dt = \int_{\mathbb{R}} \mathbf{1}_{[-a,a]}(t) p'(t) \left(\int_{-a}^{a} \mathbf{1}_{]-\infty,t]}(s)\, dE_{x,y}(s) \right) dt$$