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Limit cycle bifurcations by perturbing a class of integrable systems with a polycycle ☆

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ABSTRACT

In this paper, we deal with the problem of limit cycle bifurcation near a 2-polycycle or 3-polycycle for a class of integrable systems by using the first order Melnikov function. We first get the formal expansion of the Melnikov function corresponding to the heteroclinic loop and then give some computational formulas for the first coefficients of the expansion. Based on the coefficients, we obtain a lower bound for the maximal number of limit cycles near the polycycle. As an application of our main results, we consider quadratic integrable polynomial systems, obtaining at least two limit cycles.

1. Introduction

Many works have been done on limit cycle bifurcations for near-Hamiltonian systems with a family of periodic orbits (see [1–6,8–12,14] for example). However, in recent years, there were some works on limit cycle bifurcations by perturbing non-Hamilton integrable systems (see [7,13,15–17] for example). As we know, the main tool for studying the bifurcation problem of limit cycles is to use the first order Melnikov function. By investigating the number of zeros of the function, we can find a lower bound for the maximal number of limit cycles inside the family of systems with parameters. An efficient method for finding zeros of the function is to study the expansions of the function near some values of the Hamiltonian function corresponding to a center, homoclinic loop or a polycycle, see the survey article [4]. However, from [1,7,15] we see that there exist such limit cycles near a heteroclinic loop that cannot be obtained by zeros of the Melnikov function. This kind of limit cycles are said to be alien limit cycles.

In this paper, we consider a C^∞ system of the form

$$\dot{x} = F(x)y + \epsilon p(x, y), \qquad \dot{y} = G(x) + R(x)y^2 + \epsilon q(x, y) \tag{1.1}$$

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with

$$p(x,y)=\sum_{j=0}^{n}\hat{p}_j(x)y^j,\qquad q(x,y)=\sum_{j=0}^{n}\hat{q}_j(x)y^j,\tag{1.2}$$

where $F(x)$, $G(x)$, $R(x)$, $\hat{p}_j(x)$, and $\hat{q}_j(x)$ are C^∞ functions, and $0\leqslant\epsilon\ll 1$. For $\epsilon=0$, (1.1) becomes the unperturbed system

$$\dot{x}=F(x)y,\qquad \dot{y}=G(x)+R(x)y^2.\tag{1.3}$$

Suppose the functions F, G, R satisfy

$$F(0)=0,\qquad F(x)=xF_1(x),\qquad F_1(0)>0,\qquad G(0)>0,\qquad R(0)<0.\tag{1.4}$$

Under the condition (1.4), it is not difficult to see that system (1.3) has two hyperbolic saddles $(0,\pm y_0)$ with $y_0=\sqrt{-\frac{G(0)}{R(0)}}>0$, lying on the invariant line $x=0$. If, moreover, there exists $x_1>0$ such that

$$F_1(x_1)>0,\qquad G'(x_1)>0,\qquad G(x_1)=0,\tag{1.5}$$

then system (1.3) has a third hyperbolic saddle $(x_1,0)$.

We will suppose (1.3) has a 2-polycycle with saddles $(0,\pm y_0)$ or a 3-polycycle with saddles $(0,\pm y_0)$ and $(x_1,0)$. Note that the polycycle contains a connection lying on the y-axis. We will see that (1.3) has an integral factor which is not well-defined at $x=0$. In this case, the bifurcation problem of limit cycles near the polycycle has not been studied so far to our knowledge. Our goal in this paper is to study the expansions of the first order Melnikov function near the polycycle and then to study the number of limit cycles of (1.1) based on the expansions. In this way, we obtain a lower bound of the maximal number of limit cycles for the perturbed system.

We organize the paper as follows. In Section 2, we obtain the expansion of the first order Melnikov function near a 2-polycycle. In Section 3, we obtain the expansion of the function near a 3-polycycle. In Section 4, we deal with the bifurcation of limit cycles from a 2-polycycle or 3-polycycle. In Section 5, we present some applications of our main results obtained in Sections 2 and 3 to quadratic polynomial systems.

2. The expansion near a 2-polycycle

We first study the unperturbed system (1.3). The following lemma shows that (1.3) is integrable.

Lemma 2.1. *Let (1.4) be satisfied. Then system (1.3) has an integrating factor of the form* $\mu(x)=x^\alpha\mu_0(x)$ *with* $\alpha=-\frac{2R(0)}{F_1(0)}-1$, $\mu_0(x)\in C^\infty$, *and* $\mu_0(0)>0$, $\alpha>-1$.

Proof. Let us assume that $\mu=\mu(x)$ is an integrating factor of (1.3). Then it should satisfy

$$\big[\mu(x)F(x)y\big]_x+\big[\mu(x)\big(G(x)+R(x)y^2\big)\big]_y\equiv 0.$$

That is,

$$\frac{\mathrm{d}(\mu F)}{\mathrm{d}x}=-\frac{2R}{F}(\mu F).$$

Solving the differential equation, we obtain

$$\mu(x)=\frac{1}{F(x)}e^{-\int\frac{2R(x)}{F(x)}\,\mathrm{d}x}.$$