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Operator theoretic differences between Hardy and Dirichlet-type spaces ☆

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ABSTRACT

For $0 < p < \infty$, the Dirichlet-type space $\mathcal{D}^p_{p-1}$ consists of the analytic functions f in the unit disc $\mathbb{D}$ such that $\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1}\, dA(z) < \infty$. Motivated by operator theoretic differences between the Hardy space H^p and $\mathcal{D}^p_{p-1}$, the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

acting from one of these spaces to another is studied. In particular, it is shown, on one hand, that $T_g : \mathcal{D}^p_{p-1} \to H^p$ is bounded if and only if $g \in \mathrm{BMOA}$ when $0 < p \leqslant 2$, and, on the other hand, that this equivalence is very far from being true if $p > 2$. Those symbols g such that $T_g : \mathcal{D}^p_{p-1} \to H^q$ is bounded (or compact) when $p < q$ are also characterized. Moreover, the best known sufficient L^∞-type condition for a positive Borel measure μ on $\mathbb{D}$ to be a p-Carleson measure for $\mathcal{D}^p_{p-1}$, $p > 2$, is significantly relaxed, and the established result is shown to be sharp in a very strong sense.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z\colon\ |z| < 1\}$ of the complex plane $\mathbb{C}$. Let $\mathbb{T}$ be the boundary of $\mathbb{D}$. The *Carleson square* associated with an interval $I \subset \mathbb{T}$ is the set

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$S(I)=\{re^{it}\colon\ e^{it}\in I,\ 1-|I|\leqslant r<1\}$, where $|E|$ denotes the normalized Lebesgue measure of the set $E\subset\mathbb{T}$. For our purposes it is also convenient to define for each $a\in\mathbb{D}\setminus\{0\}$ the interval $I_a=\{e^{i\theta}\colon\ |\arg(ae^{-i\theta})|\leqslant\pi(1-|a|)\}$, and denote $S(a)=S(I_a)$. For $0<p\leqslant\infty$, the *Hardy space* H^p consists of the functions $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}=\lim_{r\to1^-}M_p(r,f)<\infty,$$

where

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\,d\theta\right)^{\frac{1}{p}},\quad 0<p<\infty,$$

and

$$M_\infty(r,f)=\max_{0\leqslant\theta\leqslant2\pi}\left|f\left(re^{i\theta}\right)\right|.$$

For the theory of the Hardy spaces, see [9,12].

For $0<p<\infty$ and $-1<\alpha<\infty$, the *Dirichlet space* $\mathcal{D}^p_\alpha$ consists of all $f\in\mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha}=\int_{\mathbb{D}}\left|f'(z)\right|^p\left(1-|z|^2\right)^\alpha\,dA(z)+\left|f(0)\right|^p<\infty,$$

where $dA(z)=\frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$.

The purpose of this study is to underline operator theoretic differences between the closely related spaces $\mathcal{D}^p_{p-1}$ and H^p. Before going to that, it is appropriate to recall inclusion relations between these spaces. The classical Littlewood–Paley formula implies $D^2_1=H^2$. Moreover, it is well known [10,17] that

$$\mathcal{D}^p_{p-1}\subsetneq H^p,\quad 0<p<2,\tag{1.1}$$

and

$$H^p\subsetneq\mathcal{D}^p_{p-1},\quad 2<p<\infty.\tag{1.2}$$

It is also worth mentioning that there are no inclusion relations between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$ when $p\neq q$ [14].

A natural way to illustrate differences between two given spaces is to consider classical operators acting on them. For example, if $0<p<2$, then the behavior of the *composition operator* $C_\varphi(f)=f\circ\varphi$ reveals that $\mathcal{D}^p_{p-1}$ is in a sense a much smaller space than H^p. Namely, it follows from Littlewood's subordination theorem that $C_\varphi:H^p\to H^p$ is bounded for each $0<p<\infty$ and all analytic self-maps φ of $\mathbb{D}$, but in contrast to this, there are symbols φ which induce unbounded operators $C_\varphi:\mathcal{D}^p_{p-1}\to\mathcal{D}^p_{p-1}$ when $0<p<2$ [8, Theorem 1.1(b)]. As in the case of Hardy spaces, any composition operator $C_\varphi:\mathcal{D}^p_{p-1}\to\mathcal{D}^p_{p-1}$ is bounded when $2\leqslant p<\infty$.

There are operators which do not distinguish between $\mathcal{D}^p_{p-1}$ and H^p. For a given $g\in\mathcal{H}(\mathbb{D})$, the *generalized Hilbert operator* $\mathcal{H}_g$ is defined by

$$\mathcal{H}_g(f)(z)=\int_0^1 f(t)g'(tz)\,dt,\tag{1.3}$$