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Numerical range for random matrices

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A B S T R A C T

We analyze the numerical range of high-dimensional random matrices, obtaining limit results and corresponding quantitative estimates in the non-limit case. For a large class of random matrices their numerical range is shown to converge to a disc. In particular, numerical range of complex Ginibre matrix almost surely converges to the disk of radius $\sqrt{2}$. Since the spectrum of non-hermitian random matrices from the Ginibre ensemble lives asymptotically in a neighborhood of the unit disk, it follows that the outer belt of width $\sqrt{2}-1$ containing no eigenvalues can be seen as a quantification the non-normality of the complex Ginibre random matrix. We also show that the numerical range of upper triangular Gaussian matrices converges to the same disk of radius $\sqrt{2}$, while all eigenvalues are equal to zero and we prove that the operator norm of such matrices converges to $\sqrt{2e}$.

1. Introduction

In this paper we are interested in the numerical range of large random matrices. In general, *the numerical range* (also called *the field of values*) of an $N \times N$ matrix is defined as $W(X) = \{(Xy, y)\colon \ \|y\|_2 = 1\}$

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(see e.g. [18,22,24]). This notion was introduced almost a century ago and it is known by the celebrated Toeplitz–Hausdorff theorem [21,38] that $W(X)$ is a compact convex set in $\mathbb{C}$. A common convention to denote the numerical range by $W(X)$ goes back to the German term "Wertevorrat" used by Hausdorff.

For any $N \times N$ matrix X its numerical range $W(X)$ clearly contains all its eigenvalues λ_i, $i \leqslant N$. If X is normal, that is $XX^* = X^*X$, then its numerical range is equal to the convex hull of its spectrum, $W(X) = \Gamma(X) := \operatorname{conv}(\lambda_1, \dots, \lambda_N)$. The converse is valid if and only if $N \leqslant 4$ [32,23].

For a non-normal matrix X its numerical range is typically larger than $\Gamma(X)$ even in the case $N = 2$. For example, consider the Jordan matrix of order two,

$$J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then both eigenvalues of J_2 are equal to zero, while $W(J_2)$ forms a disk $D(0, 1/2)$.

We shall now turn our attention to numerical range of random matrices. Let G_N be a complex random matrix of order N from the *Ginibre ensemble*, that is an $N \times N$ matrix with i.i.d. centered complex normal entries of variance $1/N$. It is known that the limiting spectral distribution μ_N converges to the uniform distribution on the unit disk with probability one (cf. [5,15–17,36,37]). It is also known that the operator norm goes to 2 with probability one. This is directly related to the fact that the level density of the Wishart matrix $G_N G_N^*$ is asymptotically described by the Marchenko–Pastur law, supported on $[0, 4]$, and the squared largest singular value of G_N goes to 4 ([19], see also [14] for the real case).

As the complex Ginibre matrix G_N is generically non-normal, the support Γ of its spectrum is typically smaller than the numerical range W. Our results imply that the ratio between the area of $W(G_N)$ and $\Gamma(G_N)$ converges to 2 with probability one. Moreover, in the case of strictly upper triangular matrix T_N with Gaussian entries (see below for precise definitions) we have that the area of $W(T_N)$ converges to 2, while clearly $\Gamma(T_N) = \{0\}$.

The numerical range of a matrix X of size N can be considered as a projection of the set of density matrices of size N,

$$Q_N = \{\varrho\colon\ \varrho = \varrho^*,\ \varrho \geqslant 0,\ \operatorname{Tr} \varrho = 1\},$$

onto a plane, where this projection is given by the (real) linear map $\rho \mapsto \operatorname{Tr} \varrho X$. More precisely, for any matrix X of size N there exists a real affine rank 2 projection P of the set Q_N, whose image is congruent to the numerical range $W(X)$ [10].

Thus our results on numerical range of random matrices contribute to the understanding of the geometry of the convex set of quantum mixed states for large N.

Let d_H denote the Hausdorff distance. Our main result, Theorem 4.1, states the following:

If random matrices X_N of order N satisfy for every real θ

$$\lim_{N\to\infty} \left\| \operatorname{Re}\left(e^{i\theta} X_N\right) \right\| = R$$

then with probability one

$$\lim_{N\to\infty} d_H\left(W(X_N), D(0, R)\right) = 0.$$

We apply this theorem to a large class of random matrices. Namely, let $x_{i,i}$, $i \geqslant 1$, be i.i.d. complex random variables with finite second moment, $x_{i,j}$, $i \neq j$, be i.i.d. centered complex random variables with finite fourth moment, and all these variables are independent. Assume $\mathbb{E}|x_{1,2}|^2 = \lambda^2$ for some $\lambda > 0$. Let