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Asymptotics for multiple Meixner polynomials

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ABSTRACT

We study the asymptotic behavior of multiple Meixner polynomials of first and second kind, respectively [6]. We use an algebraic function formulation for the solution of the equilibrium problem with constraint to describe their zero distribution. Moreover, analyzing the limiting behavior of the coefficients of the recurrence relations for Multiple Meixner polynomials we obtain the main term of their asymptotics.

1. Introduction

This paper deals with the asymptotics of polynomial sequences (P_n) defined by orthogonality relations with respect to a discrete measure μ (with finite moments)

$$\mu = \sum_{k=0}^{N} \rho(x)\delta_{x_k}, \quad \rho(x_k) = \rho_k > 0,\ x_k \in \mathbb{R},\ N \in \mathbb{N} \cup \{+\infty\}, \tag{1.1}$$

which is a linear combination of Dirac measures at the points $x_0, \ldots, x_N$. By $\mathbb{N}$ we denote the set of all nonnegative integers.

The asymptotic theory of orthogonal polynomials with respect to discrete measures is not so widely developed as for absolutely continuous measures of orthogonality. It can be explained because the distribution of zeros of the polynomials orthogonal with respect to a discrete measure has an extra constraint. Indeed, between two neighboring mass points of the measure of orthogonality it cannot be situated more than one zero of the orthogonal polynomials (this fact follows from the interlacing property of orthogonal polynomials). This constraint plays an important role in the logarithmic potential description of the limiting zero distribution measure as well as of the main term of asymptotics.

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The logarithmic potential

$$\mathcal{P}^{\nu}(z)=-\int \ln|z-t|\,d\nu(t),$$

for probability measure $\nu_{P_n}(t)$ (*zero counting measure*), equally distributed in the zeros of polynomial P_n, is given by

$$\mathcal{P}^{\nu_{P_n}}=-\frac{1}{n}\ln|P_n|.$$

Thus, the potential of a limiting (when $n\to\infty$) zero distribution measure λ defines an *exponent of the main term* of the asymptotics for the polynomial sequence (P_n). Such type of asymptotics are also called *weak asymptotics*. For the description of this limiting measure it is useful to formulate a minimization problem for the energy of the logarithmic potential. It is clear, that the constraint of the zero counting measure (which we pointed out above) leads to a problem of energy minimization in the class of probabilistic measures constrained by the weak limit of the mass points of the zero counting measure. For discrete orthogonal polynomials such an approach was suggested by Rakhmanov in [21] and extended later by Dragnev and Saff in [9]. For more details on the topic we refer to monographs [20] and [22] for extremal and equilibrium problem of logarithmic potential with application to the weak asymptotics of polynomial sequences as well as to monograph [7] for discrete orthogonality and constrained equilibrium.

The content of this paper starts with the study of weak asymptotics for monic polynomial sequences (P_n) orthogonal with respect to discrete measures (1.1). In particular, the polynomial sequence orthogonal with respect to negative binomial distribution (Pascal distribution) on $\mathbb{N}$, i.e.

$$\rho_k=\frac{(\beta)_k}{k!}c^k\quad(\beta>0,\ 0<c<1),\qquad x_k=k,\quad k\in\mathbb{N}, \tag{1.2}$$

is considered. These orthogonal polynomials (denoted by $M_n(x;\beta,c)$) are called *classical Meixner polynomials* [17]. They satisfy the orthogonality conditions

$$\sum_{k=0}^{+\infty}M_n(k;\beta,c)(-k)_j\frac{(\beta)_k}{k!}c^k=0,\quad j=0,1,\dots,n-1, \tag{1.3}$$

where $(-k)_j=-k(1-k)\cdots(j-k-1)$, $j\in\mathbb{N}$, and $(k)_0=1$, is the Pochhammer symbol (for the asymptotics of these classical polynomials $M_n(x;\beta,c)$, we refer to [27] and [29]). However, our main goal is the asymptotic analysis of *multiple Meixner polynomials*, which constitute a generalization of the aforementioned classical Meixner polynomials. For these polynomials the orthogonality conditions are considered with respect to a collection of Meixner weights (1.2) with different parameters β or c. The study of these polynomials (among others classical discrete multiple orthogonal polynomials) was initiated in [6].

The structure of the paper is as follows. Section 1 introduces the context and the background materials compressed in four subsections. Indeed, Section 1.1 is devoted to the notion of multiple orthogonal polynomials. In Section 1.2, some general information about recurrence relations for multiple orthogonal polynomials is given. The last two subsections are devoted to some formal properties of the multiple Meixner polynomials obtained in [6]. We recall that there are two kinds of multiple Meixner polynomials. The first kind corresponds (see Section 1.3) to simultaneous orthogonality conditions with respect to discrete weights formed by (1.2) with various c_i, $i=1,\dots,r$. The second kind polynomials (see Section 1.4) appear when the collection of discrete weights is formed by (1.2) with various β_i, $i=1,\dots,r$. The explicit expressions for the coefficients of the recurrence relations of multiple Meixner polynomials are the main outcome from Sections 1.3 and 1.4 – used in the sequel.

In Section 2, we state the logarithmic potential equilibrium problems for the description of the weak asymptotics of Meixner and multiple Meixner polynomials. Moreover, the solutions of these problems by means of certain algebraic functions are found. The procedure for solving such type of equilibrium problems is inspired in [2] (see also earlier references therein).

Lastly, in Section 3, by starting from the coefficient of the recurrence relations, we obtain the main term of the asymptotics of multiple Meixner polynomials (the weak asymptotics) and then we check the connection of this term with the equilibrium problem from Section 2. Indeed, Theorems 3.2 and 3.3 are the main results of the present paper. Furthermore, the spectral curves for the multiple Meixner polynomials (of the first and second kind) and their connection with the recurrence relations and vector potential equilibrium with external field and constraint constitute the main outcome of the present approach.

Finally, we would like to highlight that in [23] the weak asymptotics of multiple Meixner polynomials of the second kind were studied, but using a different approach in which the integral representation for the polynomials and the saddle point method were applied as main techniques.

1.1. Multiple orthogonal polynomials (type II)

Below we summarize some results needed for the sequel.