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Bifurcation of limit cycles by perturbing a class of hyper-elliptic Hamiltonian systems of degree five ☆

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ABSTRACT

In this paper, we investigate a class of hyper-elliptic Hamiltonian systems of degree five under the polynomial perturbation of degree $m+1$. First, we study the number of different phase portraits of the unperturbed system when it has a class of family of periodic orbits and prove that the number is 40. Then, we consider the limit cycle bifurcations and obtain some new results on the lower bound of the maximal number of limit cycles for these systems.

1. Introduction and main results

The second part of the Hilbert's 16th problem asks to find an upper bound for the number of limit cycles and their relative locations for polynomial vector fields of degree n. See [12]. Although the problem is still far from being completely solved, the research on this problem has made great progress with significant contributions to the development of modern mathematics, see [1–5,7,8,11,9,10,13–16,18,17,21,19,20,22]. As we have seen, many researchers have been investigating the following Liénard system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - \varepsilon y f(x), \tag{1.1}$$

where $\varepsilon > 0$ is sufficiently small, $f(x)$ and $g(x)$ are polynomials of degree m and n, respectively. For $\varepsilon = 0$, the above system reduces to

$$\dot{x} = y, \qquad \dot{y} = -g(x), \tag{1.2}$$

which is a Hamiltonian system with the Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 + G(x), \quad G(x) = \int_0^x g(s)\,ds, \tag{1.3}$$

where G is a polynomial in x of degree $n+1$. The level curves of H are rational for $n = 0,1$, elliptic for $n = 2,3$ and hyper-elliptic for $n \geqslant 4$. Let $Z(n)$ denote the number of different phase portraits of (1.2) when it has at least a family of period orbits. Obviously, $Z(0) = 0$, $Z(1) = Z(2) = 1$. The maximal number of limit cycles of (1.1) is denoted by $H(m,n)$. There are many papers to study $Z(n)$ and $H(m,n)$. We describe briefly as follows:

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(i) For $n=1$, Blows and Lloyd [1] proved $H(m,1)\geqslant[\frac{m}{2}]$;
(ii) For $n=2$, Han [5,6] obtained $H(m,2)=[\frac{2m+1}{3}]$ with $m\geqslant 2$;
(iii) For $n=3$, from Dumortier and Li [3] one can find that $Z(3)=6$; Christopher and Lynch [2] proved that $H(m,3)\geqslant 2[\frac{3m+6}{8}]$ with $2\leqslant m\leqslant 50$; Then, Yang et al. [21] achieved that $H(m,3)\geqslant[\frac{3m+14}{4}]$ with $m=3,4,5,6,7,8$; Yang and Han [19] gave an estimation for the lower bound of $H(m,3)\geqslant m+2-[\frac{m+1}{4}]$, $6\leqslant m\leqslant 22$;
(iv) For $n=4$, Gavrilov and Iliev [4] or Xiao [16] obtained $Z(4)=11$; Christopher and Lynch [2] gave $H(9,4)\geqslant 9$. And then, Yu and Han [22] proved that $H(m,4)\geqslant m$, $m=10,11,12,13,14$; Further, Han et al. [8] obtained $H(m,4)\geqslant m+3$, $m=2,3,5,6,7,8$ and $H(4,4)\geqslant 6$; Then, Yang and Han [20] gave that $H(m,4)\geqslant m+4-[\frac{m+1}{5}]$, $m=9,\dots,18$.
(v) For more general results, Han and Romanovski [7] proved that $H(m,4)\geqslant H(m,3)\geqslant 2[\frac{m-1}{4}]+[\frac{m-1}{2}]$ for $m\geqslant 3$, $H(m,6)\geqslant H(m,5)\geqslant 2[\frac{m-1}{3}]+[\frac{m-1}{2}]$ with $m\geqslant 5$, $H(m,7)\geqslant \frac{3}{2}m-9$ for $m\geqslant 7$.

This paper mainly discusses $Z(5)$ and $H(m,5)$. Then, $g(x)$ is restricted to a polynomial of degree 5. Hence, it can be rewritten as

$$g(x)=\alpha(x-a)(x-b)(x-c)(x-d)(x-e) \tag{1.4}$$

where $\alpha>0$ or $\alpha<0$, a,b,c,d,e must satisfy one of the following 21 cases:

1. $\bar a=b$, $\bar c=d\in\mathbb{C}\setminus\mathbb{R}$, $e\in\mathbb{R}$;
2. $\bar a=b\in\mathbb{C}\setminus\mathbb{R}$, $c,d,e\in\mathbb{R}$, $c<d<e$;
3. $\bar a=b\in\mathbb{C}\setminus\mathbb{R}$, $c,d,e\in\mathbb{R}$, $c=d<e$;
4. $\bar a=b\in\mathbb{C}\setminus\mathbb{R}$, $c,d,e\in\mathbb{R}$, $c=d=e$;
5. $a,b,c,d,e\in\mathbb{R}$, $a<b<c<d<e$;
6. $a,b,c,d,e\in\mathbb{R}$, $a=b<c<d<e$;
7. $a,b,c,d,e\in\mathbb{R}$, $a<b=c<d<e$;
8. $a,b,c,d,e\in\mathbb{R}$, $a=b=c<d<e$;
9. $a,b,c,d,e\in\mathbb{R}$, $a<b=c=d<e$;
10. $a,b,c,d,e\in\mathbb{R}$, $a=b=c=d<e$;
11. $a,b,c,d,e\in\mathbb{R}$, $a=b=c=d=e$;
12. $a,b,c,d,e\in\mathbb{R}$, $a=b<c=d<e$;
13. $a,b,c,d,e\in\mathbb{R}$, $a=b<c<d=e$;
14. $a,b,c,d,e\in\mathbb{R}$, $a=b=c<d=e$;
15. $\bar a=b\in\mathbb{C}\setminus\mathbb{R}$, $c,d,e\in\mathbb{R}$, $c<d=e$;
16. $a,b,c,d,e\in\mathbb{R}$, $a<b<c<d=e$;
17. $a,b,c,d,e\in\mathbb{R}$, $a<b<c=d<e$;
18. $a,b,c,d,e\in\mathbb{R}$, $a<b<c=d=e$;
19. $a,b,c,d,e\in\mathbb{R}$, $a<b=c=d=e$;
20. $a,b,c,d,e\in\mathbb{R}$, $a<b=c<d=e$;
21. $a,b,c,d,e\in\mathbb{R}$, $a=b<c=d=e$.

It is easy to see that cases 1, 4 and 11 have one real zero, cases 3, 10, 14, 15, 19 and 21 have two different real zeros, cases 2, 8, 9, 12, 13, 18 and 20 have three different real zeros, cases 6, 7, 16 and 17 have four different real zeros, and case 5 has five different real zeros. Make a transformation and time scaling

$$x_1=-x,\qquad y_1=y,\qquad t\to -t. \tag{1.5}$$

Then, system (1.2) becomes

$$\dot x_1=y_1,\qquad \dot y_1=g(-x_1)=-\alpha(x_1+a)(x_1+b)(x_1+c)(x_1+d)(x_1+e)$$

which implies that cases 15, 16, 17, 18, 19, 20, 21 are equivalent to cases 3, 6, 7, 8, 10, 12, 14, respectively. Thus, we only need to investigate cases 1–14, and easily obtain Table 1.

We remark that in Table 1:

Npo := No periodic orbits,	Ec := Elementary center,	Hs := Hyperbolic saddle,
Cuk := Cusp of order k,	Nck := Nilpotent center of order k,	Nsk := Nilpotent saddle of order k.

Using (1.5), one can see that $G(b)>G(d)$, $G(a)>G(d)$, $G(c)>G(e)$, $G(a)>G(c)=G(e)$, $G(a)<G(c)=G(e)$, $G(a)<G(c)<G(e)$, $G(a)>G(e)>G(c)$ and $G(a)>G(e)$ are equivalent to $G(b)<G(d)$, $G(a)<G(d)$, $G(c)<G(e)$, $G(a)=G(c)<G(e)$, $G(a)=G(c)>G(e)$, $G(a)>G(c)>G(e)$, $G(c)<G(a)<G(e)$ and $G(a)<G(e)$ in Figs. 1.1(5), 1.1(14), 1.2(1), 1.2(4), 1.2(5), 1.2(7), 1.2(9) and 1.2(21), respectively. Thus, we obtain

Theorem 1.1. *There are* 40 *different cases for the phase portraits of system* (1.2) *under* (1.4)*. That is to say* $Z(5)=40$.

Now, we turn to consider bifurcations of limit cycles on system (1.1). Kazemi et al. [14] studied system (1.1) in the case 8 of Fig. 1.2(19) with $g(x)=-x(x-\frac{1}{2})(x-1)^3$ and obtained that $H(4,5)\geqslant 3$. Xu and Li [18] considered system (1.1) in the case 5 of Fig. 1.2(6) by taking $g(x)=-(x+1)(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)$ and proved that $H(2,5)\geqslant 3$, $H(4,5)\geqslant 5$, $H(6,5)\geqslant 10$, $H(8,5)\geqslant 10$. Xiong and Zhong [17] investigated (1.1) in the case 5 of Fig. 1.1(6) with $g(x)=(x+\sqrt{2})(x+\frac{\sqrt{2}}{2})x(x-\frac{\sqrt{2}}{2})(x-\sqrt{2})$ and achieved that $H(2,5)\geqslant 5$, $H(4,5)\geqslant 6$, $H(8,5)\geqslant 12$. In this paper, we consider system (1.1) in the case 6 of Fig. 1.1(8). Without lose of generality, we can take $a=b=0$. Then (1.1) becomes