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Embeddable quantum homogeneous spaces ☆

Paweł Kasprzak [a,b], Piotr M. Sołtan [a,*]

[a] Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Poland
[b] Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland

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ABSTRACT

We discuss various notions generalizing the concept of a homogeneous space to the setting of locally compact quantum groups. On the von Neumann algebra level we recall an interesting duality for such objects studied earlier by M. Izumi, R. Longo, S. Popa for compact Kac algebras and by M. Enock in the general case of locally compact quantum groups. A definition of a quantum homogeneous space is proposed along the lines of the pioneering work of Vaes on induction and imprimitivity for locally compact quantum groups. The concept of an embeddable quantum homogeneous space is selected and discussed in detail as it seems to be the natural candidate for the quantum analog of classical homogeneous spaces. Among various examples we single out the quantum analog of the quotient of the Cartesian product of a quantum group with itself by the diagonal subgroup, analogs of quotients by compact subgroups as well as quantum analogs of trivial principal bundles. The former turns out to be an interesting application of the duality mentioned above.

1. Introduction

The notion of a homogeneous space of a locally compact group is of fundamental importance in many branches of mathematics. The non-commutative geometric generalization of the theory of locally compact groups was enriched greatly by the paper of S. Vaes [35], in which the notion of a closed subgroup and, more importantly a quantum homogeneous space was thoroughly discussed (alongside many other developments). For compact quantum groups these notions were already developed in the PhD thesis of P. Podleś [20] and later published in [21]. It was shown in that paper that quantum groups often have fewer subgroups than one would expect. This was, in particular, proved for the quantum SU(2) group which is a deformation of the classical SU(2), yet whose list of subgroups is dramatically shorter than that of the classical SU(2). Thus Podleś realized that some quantum homogeneous spaces did not come from quantum subgroups. It led him to introduce the notions of

- ► homogeneous space,
- ► embeddable homogeneous space,
- ► quotient homogeneous space

in the compact quantum group context and he proved that these three classes are consecutively strictly smaller. Moreover in his work on quantum spheres [19] he showed that by allowing (quantum) homogeneous spaces which are not of the

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* Corresponding author.
E-mail addresses: pawel.kasprzak@fuw.edu.pl (P. Kasprzak), piotr.soltan@fuw.edu.pl (P.M. Sołtan).

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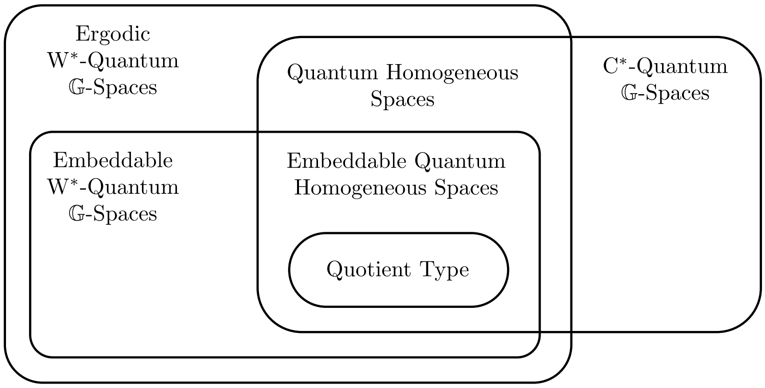

Fig. 1. Schematic presentation of relations between types of quantum group actions.

quotient type we reveal a wealth of new examples of interesting quantum spaces. The three different classes of (quantum) homogeneous spaces mentioned above are reduced to usual homogeneous spaces provided the quantum group *and* the homogeneous space are classical. It is important to note that the first (broadest) class allows non-commutative homogeneous spaces for classical groups. However (quantum) homogeneous spaces of the second and third class for a classical group are necessarily classical.

The class of *embeddable* homogeneous spaces was defined by Podleś as containing quantum spaces $\mathbb{X}$ with an action of a compact quantum group $\mathbb{G}$ which can be realized inside $C(\mathbb{G})$ by the comultiplication. In other words he considered coideals in $C(\mathbb{G})$ (cf. Theorem 4.5(2) and Proposition 3.1). The classical correspondence between closed subgroups and coideals has by now received more attention from researchers in the theory of quantum groups. We would like to point out an interesting approach, due to several authors, using *idempotent states* (see e.g. [6,7,28,26]). The case of co-amenable compact quantum groups was treated from this point of view in [7, Theorem 4.1]. A similar result for unimodular co-amenable locally compact quantum groups can be found in [28, Theorem 3.5].

This paper is devoted to generalization of the notion of embeddable homogeneous spaces for compact quantum groups to the context of locally compact quantum groups of Kustermans and Vaes [12,14,17]. We will follow the path begun by Vaes in [35], where he dealt with the generalization to the non-compact case of the quotient construction considered by Podleś.

The task seems quite a lot more complicated than for compact quantum groups. We encounter many different classes of objects related to a given locally compact quantum group. We will introduce all these notions in Sections 2, 3 and 4. Nevertheless, before giving precise definitions, we wish to present a diagram describing (informally) relations between the various concepts which will be dealt with in the paper. This is done in Fig. 1.

The intersection between (ergodic) W*-quantum $\mathbb{G}$-spaces and C*-quantum $\mathbb{G}$-spaces in Fig. 1 should not be understood literally. It represents the class of W*-quantum $\mathbb{G}$-spaces for which there exists a suitably compatible "C*-version" (see Definition 4.1).

One of the reasons for considering so many different objects is that some new constructions are natural and relatively easy to perform on one level (e.g. the W*-level) and seem quite complicated if not impossible on other levels (e.g. the C*-level). This is exemplified in particular in Section 3 where we discuss a very satisfying duality (in the spirit of [31]) between embeddable W*-quantum $\mathbb{G}$-spaces which was introduced for compact Kac algebras in [10, Section 4] and for locally compact quantum groups in [5, Section 3]. This duality produces for each embeddable W*-quantum $\mathbb{G}$-space $\mathbb{X}$ an embeddable W*-quantum $\widehat{\mathbb{G}}$-space which we denote by $\widetilde{\mathbb{X}}$ and call the *co-dual* of $\mathbb{X}$.[1] We include the proof that the second co-dual $\widetilde{\widetilde{\mathbb{X}}}$ is equal to $\mathbb{X}$ (this is a true equality, not isomorphism — the price we pay for this is that our objects come with a particular embedding into operators on appropriate Hilbert spaces). A very similar result is contained in [5, Théorème 3.3], but in a slightly different context as the duality discussed there maps right co-ideals to left ones. This duality, when restricted to a classical group $\mathbb{G}$, coincides with the well-known Takesaki–Tatsuuma duality between invariant subalgebras of $L^\infty(\mathbb{G})$ and closed subgroups of $\mathbb{G}$ (see [31]). Since our duality gives rise to the duality between invariant subalgebras of $L^\infty(\mathbb{G})$ and $L^\infty(\widehat{\mathbb{G}})$ it should be stressed that it is the co-commutativity of $\widehat{\mathbb{G}}$ that forces all invariant subalgebras of $L^\infty(\widehat{\mathbb{G}})$ to be of the form $L^\infty(\widehat{\mathbb{H}})$ for certain closed quantum subgroup $\mathbb{H} \subset \mathbb{G}$. In the case of non-classical group $\mathbb{G}$ we get a fully symmetric picture of the duality that links embeddable W*-quantum $\mathbb{G}$-spaces and embeddable W*-quantum $\widehat{\mathbb{G}}$-spaces.

The necessary material on quantum groups and the operator algebra techniques we use can be found in [12,37]. The research presented in this paper is based strongly on the paper of Vaes [35], the work on quantum subgroups [4] (which, in turn, follows [18] closely) and [11]. Some important definitions and concepts will be touched upon in Section 2.

[1] The reason for the term "co-dual" in favor of "dual" stems from the fact that the co-dual of the quantum group itself treated as a homogeneous space is not its dual as a quantum group.