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Structural characterization of hybrid equations with tractability index at most two

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ABSTRACT

In circuit simulation, differential-algebraic equations (DAEs) to be solved numerically are obtained through circuit analysis methods. While the modified nodal analysis (MNA) is the most popular method, the hybrid analysis has been shown to have inherent advantage over MNA in terms of index, which is a measure of the numerical difficulty of DAEs. A recent paper of Iwata, Takamatsu, and Tischendorf (2012) has characterized the network structure that leads to DAEs with tractability index zero and one in the hybrid analysis. As a sequel, this paper gives a necessary condition for the hybrid equations to have tractability index at most two. Moreover, we show that this condition is also sufficient for linear time-invariant circuits if dependent sources satisfy a genericity assumption. Since commonly-used circuits fulfill the condition, it is ensured that the hybrid analysis results in a DAE with tractability index at most two in most cases. By combining our results with the previous one, we obtain criteria for tractability index zero/one/two.

1. Introduction

Mathematical modeling and numerical computation are of great importance in circuit simulation. Circuits are described by *differential-algebraic equations* (*DAEs*), which consist of algebraic equations and differential operations. The numerical difficulty of DAEs is measured by the *index*. In general, the higher the index is, the more difficult it is to solve the DAE.

Numerical computation step for DAEs has been extensively studied. For example, Gear [6] proposed the backward difference formulae, which were implemented in the DASSL code by Petzold (cf. [3]). Hairer and Wanner [8] implemented an implicit Runge–Kutta method in their RADAU5 code. These methods are applicable to DAEs with low index, and more general methods for high index DAEs have been developed recently (see [12, Chapter 8]).

While many DAE solvers have been implemented, modeling step is very critical to accuracy of numerical solutions, because the difficulty of solving a DAE increases with its index. In circuit simulation, the most popular method in modeling step is to apply the *modified nodal analysis* (*MNA*). It is shown in [5] that the *tractability index* of a DAE obtained by applying MNA to nonlinear time-varying circuits that may contain a large class of dependent sources does not exceed two. However, for a circuit with dependent sources which are not included in this class, MNA sometimes leads to a DAE with index greater than two.

Fig. 1 depicts an example of such a circuit with index three [7]. It contains a dependent current source I controlled by the current through V. Although this pathological circuit does not appear in practice, its existence suggests that we may possibly face with DAEs with index greater than two in circuit simulation.

The wide use of MNA is attributed to an automatic setup of model equations. To put it the other way around, however, MNA has no flexibility in the modeling step. In contrast, the *hybrid analysis* has an advantage that we can choose a model

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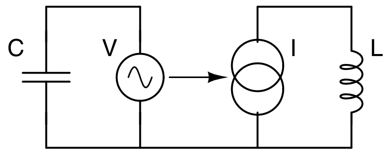

Fig. 1. A circuit with a dependent current source I, where $\rightarrow$ means that I is controlled by the current through V.

description which reduces the numerical difficulties. In the hybrid analysis, we select a *partition* of elements and a *reference tree* in the network. This selection determines DAEs, called the *hybrid equations*, to be solved numerically.

Recently, the index of the hybrid equations has been analyzed to make a theoretical comparison with MNA. An algorithm for finding an optimal pair of a partition and a reference tree which minimizes the index of the hybrid equations is proposed in [10] for linear time-invariant circuits that may contain dependent sources. For linear time-invariant RLC circuits, it is proved in [19] that the index of the hybrid equations is at most one. A structural characterization of circuits with index zero is also given in [19]. It is also shown that the index of the hybrid equations never exceeds the index of DAEs arising from MNA.

The results in [19] are extended to nonlinear time-varying circuits with dependent sources in [11], where the hybrid equations with tractability index zero and one are characterized by the network structure. These results enable us to check whether the tractability index of the hybrid equations is greater than one or not, but they do not provide a method to determine whether the tractability index is exactly equal to two or greater than two.

In this paper, we give a necessary condition for the hybrid equations to have tractability index at most two for nonlinear time-varying circuits. Moreover, we show that this condition is also sufficient for linear time-invariant circuits if dependent sources satisfy the genericity assumption. By combining these results with [11], we obtain criteria for tractability index zero/one/two.

The genericity assumption on dependent sources is motivated by the fact that dependent sources are inherently different from the other elements such as resistors, capacitors, and inductors. In contrast to the latter elements, dependent sources are used to describe an equivalent circuit model of an active device such as a transistor. The constitutive equations of dependent sources are determined by a voltage-amplification factor, a current gain, and so on. We assume that these values are independent parameters in the proof of a sufficient condition. We remark that this genericity assumption makes sense only for linear time-invariant circuits, but our results remain valid for nonlinear time-varying circuits unless unlucky numerical cancellations occur.

The organization of this paper is as follows. In Section 2, we describe nonlinear time-varying circuits and present the hybrid equations. Section 3 is devoted to the definition of the tractability index of DAEs. We analyze the hybrid equations in Section 4. Section 5 gives structural characterizations of the hybrid equations with index at most two. Finally, Section 6 concludes this paper.

2. Hybrid analysis of nonlinear time-varying circuits

In this section, we describe nonlinear time-varying circuits composed of resistors, inductors, capacitors, and independent/dependent voltage/current sources.

We denote the vector of currents through all branches of the circuit by $\boldsymbol{i}$, and the vector of voltages across all branches by $\boldsymbol{u}$. Let V, J, C, and L denote the sets of independent voltage sources, independent current sources, capacitors, and inductors, respectively. Dependent voltage/current sources are denoted by S_U and S_I. We define the set of resistors later.

The vectors of currents through V, J, C, L, S_U, and S_I are denoted by $\boldsymbol{i}_V$, $\boldsymbol{i}_J$, $\boldsymbol{i}_C$, $\boldsymbol{i}_L$, $\boldsymbol{i}_U$, and $\boldsymbol{i}_I$. Similarly, the vectors of voltages are denoted by $\boldsymbol{u}_V$, $\boldsymbol{u}_J$, $\boldsymbol{u}_C$, $\boldsymbol{u}_L$, $\boldsymbol{u}_U$, and $\boldsymbol{u}_I$. Independent voltage and current sources simply read as

$$\boldsymbol{u}_V = \boldsymbol{v}_s(t) \quad \text{and} \quad \boldsymbol{i}_J = \boldsymbol{j}_s(t). \tag{1}$$

We assume that the constitutive equations of capacitors and inductors are described by

$$\boldsymbol{i}_C = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}(\boldsymbol{u}_C, t) \quad \text{and} \quad \boldsymbol{u}_L = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(\boldsymbol{i}_L, t). \tag{2}$$

Dependent current sources and dependent voltage sources are modeled by

$$\boldsymbol{i}_I = \boldsymbol{j}_I(\boldsymbol{u}_C, \boldsymbol{u}_V, \boldsymbol{i}_L, \boldsymbol{i}_J, t) \quad \text{and} \quad \boldsymbol{u}_U = \boldsymbol{v}_U(\boldsymbol{u}_C, \boldsymbol{u}_V, \boldsymbol{i}_L, \boldsymbol{i}_J, t). \tag{3}$$

Such dependent current/voltage sources appear in many circuits [17, §6.2.6.3].

In order to provide constitutive equations of resistors, we describe the definition of an admissible partition. Let $\Gamma = (W, E)$ be the network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. For a consistent model description, Γ contains no cycles consisting only of independent voltage sources and no cutsets consisting only of independent current sources, where a *cutset* is a set of edges whose deletion increases the number of connected components in Γ. We denote the set of edges corresponding to independent voltage