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Normal criteria for families of meromorphic functions ☆

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ABSTRACT

By using the Nevanlinna theory, we prove some normality criteria for a family of meromorphic functions under a condition on differential polynomials generated by the members of the family.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_v\} \subset \mathcal{F}$, there exists a subsequence $\{f_{v_i}\}$ such that $\{f_{v_i}\}$ converges spherically locally uniformly in D, to a meromorphic function or ∞.

In 1989, Schwick proved:

Theorem A. *(See [7, Theorem 3.1].) Let k, n be positive integers such that $n \geqslant k+3$. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $(f^n)^{(k)}(z) \neq 1$ for all $z \in D$. Then $\mathcal{F}$ is normal on D.*

Theorem B. *(See [7, Theorem 3.2].) Let k, n be positive integers such that $n \geqslant k+1$. Let $\mathcal{F}$ be a family of holomorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $(f^n)^{(k)}(z) \neq 1$ for all $z \in D$. Then $\mathcal{F}$ is normal on D.*

The following normality criterion was established by Pang and Zalcman [6] in 1999:

Theorem C. *(See [6].) Let n and k be natural numbers and $\mathcal{F}$ be a family of holomorphic functions in a domain D all of whose zeros have multiplicity at least k. Assume that $f^n f^{(k)} - 1$ is non-vanishing for each $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

The main purpose of this paper is to establish some normality criteria for the case of more general differential polynomials. Our main results are as follows:

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Theorem 1. *Take q $(q \geqslant 1)$ distinct nonzero complex values $a_1, \dots, a_q$, and q positive integers (or $+\infty$) $\ell_1, \dots, \ell_q$. Let n be a nonnegative integer, and let $n_1, \dots, n_k, t_1, \dots, t_k$ be positive integers $(k \geqslant 1)$. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$ and for every $m \in \{1, \dots, q\}$, all zeros of $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a_m$ have multiplicity at least ℓ_m. Assume that*

(a) *$n_j \geqslant t_j$ for all $1 \leqslant j \leqslant k$, and $\ell_i \geqslant 2$ for all $1 \leqslant i \leqslant q$,*
(b) $\sum_{i=1}^q \frac{1}{\ell_i} < \frac{qn-2+\sum_{j=1}^k q(n_j-t_j)}{n+\sum_{j=1}^k (n_j+t_j)}$.

Then $\mathcal{F}$ is a normal family.

Take $q = 1$ and $\ell_1 = +\infty$, we get the following corollary of Theorem 1:

Corollary 2. *Let a be a nonzero complex value, let n be a nonnegative integer, and $n_1, \dots, n_k, t_1, \dots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a$ is nowhere vanishing on D. Assume that*

(a) *$n_j \geqslant t_j$ for all $1 \leqslant j \leqslant k$,*
(b) $n + \sum_{j=1}^k n_j \geqslant 3 + \sum_{j=1}^k t_j$.

Then $\mathcal{F}$ is normal on D.

We remark that in the case where $n \geqslant 3$, condition (a) in the above corollary implies condition (b); and in the case where $n = 0$ and $k = 1$, Corollary 2 gives Theorem A.

For the case of holomorphic functions, we shall prove the following result:

Theorem 3. *Take q $(q \geqslant 1)$ distinct nonzero complex values $a_1, \dots, a_q$, and q positive integers (or $+\infty$) $\ell_1, \dots, \ell_q$. Let n be a nonnegative integer, and let $n_1, \dots, n_k, t_1, \dots, t_k$ be positive integers $(k \geqslant 1)$. Let $\mathcal{F}$ be a family of holomorphic functions in a complex domain D such that for every $f \in \mathcal{F}$ and for every $m \in \{1, \dots, q\}$, all zeros of $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a_m$ have multiplicity at least ℓ_m. Assume that*

(a) *$n_j \geqslant t_j$ for all $1 \leqslant j \leqslant k$, and $\ell_i \geqslant 2$ for all $1 \leqslant i \leqslant q$,*
(b) $\sum_{i=1}^q \frac{1}{\ell_i} < \frac{qn-1+\sum_{j=1}^k q(n_j-t_j)}{n+\sum_{j=1}^k n_j}$.

Then $\mathcal{F}$ is a normal family.

Take $q = 1$ and $\ell_1 = +\infty$, Theorem 3 gives the following generalization of Theorem B, except for the case $n = k + 1$. So for the latter case, we add a new proof of Theorem B in Appendix A which is slightly simpler than the original one.

Corollary 4. *Let a be a nonzero complex value, let n be a nonnegative integer, and $n_1, \dots, n_k, t_1, \dots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of holomorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a$ is nowhere vanishing on D. Assume that*

(a) *$n_j \geqslant t_j$ for all $1 \leqslant j \leqslant k$,*
(b) $n + \sum_{j=1}^k n_j \geqslant 2 + \sum_{j=1}^k t_j$.

Then $\mathcal{F}$ is normal on D.

In the case where $n \geqslant 2$, condition (a) in the above corollary implies condition (b).

Remark 5. Our above results remain valid if the monomial $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)}$ is replaced by the following polynomial

$$f^n\left(f^{n_1}\right)^{(t_1)} \cdots \left(f^{n_k}\right)^{(t_k)} + \sum_I c_I f^{n_I}\left(f^{n_{1I}}\right)^{(t_{1I})} \cdots \left(f^{n_{kI}}\right)^{(t_{kI})},$$

where c_I is a holomorphic function on D, and n_I, n_{jI}, t_{jI} are nonnegative integers satisfying

$$\alpha_I := \frac{\sum_{j=1} t_{jI}}{n_I + \sum_{j=1}^k n_{jI}} < \alpha := \frac{\sum_{j=1} t_j}{n + \sum_{j=1}^k n_j}.$$