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Global existence and nonexistence of solutions for second-order nonlinear differential equations

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ABSTRACT

This paper deals with the global existence and nonexistence of solutions of the second-order nonlinear differential equation $(\varphi(x'))' + \lambda\varphi(x) = 0$ satisfying $x(0) = x_0$ and $x'(0) = x_1$, where λ is a positive parameter and $\varphi : (-\rho,\rho) \to (-\sigma,\sigma)$ with $0 < \rho \leqslant \infty$ and $0 < \sigma \leqslant \infty$ is strictly increasing odd bijective and continuous on $(-\rho,\rho)$. Necessary and sufficient conditions are obtained for the initial value problem to have a unique global solution which is oscillatory and periodic. Examples are given to illustrate our main result. Finally, a nonexistence result for the equation with a damping term is discussed as an application to our result.

1. Introduction

We consider the second-order nonlinear differential equation

$$\big(\varphi(x')\big)' + \lambda\varphi(x) = 0, \tag{1.1}$$

where λ is a positive parameter and $\varphi : (-\rho,\rho) \to (-\sigma,\sigma)$ with $0 < \rho \leqslant \infty$ and $0 < \sigma \leqslant \infty$ is strictly increasing odd bijective and continuous on $(-\rho,\rho)$.

A function $x(t)$ is said to be a *solution* of (1.1) on some interval $I \subset \mathbb{R}$, if $x(t)$ and $\varphi(x'(t))$ are continuously differentiable on I, and $x(t)$ satisfies (1.1) on I. In the case $I = \mathbb{R}$, we call it a *global solution*. A nontrivial global solution of (1.1) is said to be *oscillatory* if it has an infinite number of zeros on $\mathbb{R}$.

Let us consider the equation

$$\big(a(t)\varphi(x')\big)' + b(t)f(x) = 0, \tag{1.2}$$

where $a(t)$ and $b(t)$ are continuous and $a(t) > 0$ on some interval and $f(x)$ is a real valued continuous function on $(-\rho,\rho)$ and satisfies the signum condition $xf(x) > 0$ if $x \neq 0$. We can regard Eq. (1.2) as a general form of Eq. (1.1). Over the past one and half century, a great deal of articles have been devoted to the study of oscillation of solutions of linear, half-linear and nonlinear differential equations of the form (1.2). For example, those results can be found in [1–12,15–21] and the references cited therein.

To discuss whether or not solutions of (1.2) are oscillatory in a neighborhood of infinity, we need the global existence and the uniqueness of solutions of (1.2) satisfying

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$$x(0) = x_0, \qquad x'(0) = x_1 \tag{1.3}$$

for any $x_0, x_1 \in (-\rho, \rho)$, but in order to focus on the oscillation problem, we sometime assume that (1.2)–(1.3) has a unique global solution. However, as pointed out by Pan and Xing [13,14], the prescribed mean curvature equation $(\varphi_C(x'))' + \lambda f(x) = 0$ has a nonglobal solution, where $\rho = \infty$ and $\varphi_C(x) = x/\sqrt{1+x^2}$. On the other hand, in virtue of [7,9], it is known that the half-linear differential equation

$$\big(\varphi_p(x')\big)' + \lambda\varphi_p(x) = 0 \tag{1.4}$$

satisfying (1.3) has a unique global solution, where $\rho = \infty$ and $\varphi_p(x) = |x|^{p-1}\operatorname{sgn} x$. Since

$$\lim_{p\to 1+0} \varphi_p(x) = \varphi_1(x) = \lim_{\varepsilon\to+0} \varphi_C(x/\varepsilon),$$

the functions $\varphi_p(x)$ with $1 < p < 2$ and $\varphi_C(x)$ are similar in some sense. A natural question now arises: are the global existence and the uniqueness of solutions of

$$\big(\varphi_C(x')\big)' + \lambda\varphi_C(x) = 0 \tag{1.5}$$

guaranteed for the initial value problem? The purpose of this paper is to answer the question. To be precise, we give necessary and sufficient conditions for (1.1)–(1.3) to have a unique global solution. At the same time, we can show that the global solution and its derivative are oscillatory and periodic. Our main result is stated as follows:

Theorem 1.1. (1.1)–(1.3) *has a unique global solution if and only if*

$$\lambda\int_0^{x_0} \varphi(\xi)\,d\xi + \int_0^{\varphi(x_1)} \varphi^{-1}(\eta)\,d\eta < \min\left\{\lambda\int_0^{\rho}\varphi(\xi)\,d\xi,\ \int_0^{\sigma}\varphi^{-1}(\eta)\,d\eta\right\}. \tag{1.6}$$

The solution and its derivative are oscillatory and periodic if $(x_0, x_1) \neq (0, 0)$.

Remark 1.1. Since φ is strictly increasing on $(-\rho, \rho)$, there exists the inverse function of φ denoted by $\varphi^{-1} : (-\sigma, \sigma) \to (-\rho, \rho)$. We note that φ^{-1} is also strictly increasing odd continuous on $(-\sigma, \sigma)$.

Remark 1.2. We can regard a unique global solution of (1.1) and its derivative as generalized trigonometric functions. In fact, for $\lambda = 1$, $\rho = \infty$ and $\varphi(x) = x$, the function $x(t) = \sin t$ is a solution of (1.1) satisfying $x(0) = 0$ and $x'(0) = 1$.

This paper is organized as follows. In Section 2, we show the uniqueness of solutions of Eq. (1.1) to the initial value problem. In Section 3, we give the proof of the main result by means of time maps. In Section 4, we consider a special case and give three examples. Finally, in Section 5, using phase plane analysis and the main result, we show that under certain conditions, the nonlinear differential equation with a damping term $(\varphi(x'))' - \mu\varphi(x') + \lambda\varphi(x) = 0$ has no nontrivial global solutions, where μ is a positive parameter.

2. Uniqueness of solutions

Consider the system

$$x' = \varphi^{-1}(y), \qquad y' = -\lambda\varphi(x), \quad (x, y) \in (-\rho, \rho) \times (-\sigma, \sigma) \tag{2.1}$$

which is equivalent to (1.1). Let $y_0 = \varphi(x_1)$. Then (1.3) becomes

$$x(0) = x_0, \qquad y(0) = y_0, \quad (x_0, y_0) \in (-\rho, \rho) \times (-\sigma, \sigma). \tag{2.2}$$

If $(x(t), y(t))$ is a solution of (2.1)–(2.2) on some interval I, then it is easy to check that $(x(t), y(t))$ satisfies

$$\Phi\big(x(t)\big) + \Psi\big(y(t)\big) = \Phi(x_0) + \Psi(y_0) \tag{2.3}$$

on I, where

$$\Phi(x) = \lambda\int_0^x \varphi(\xi)\,d\xi \quad \text{for } -\rho \leqslant x \leqslant \rho,$$

$$\Psi(y) = \int_0^y \varphi^{-1}(\eta)\,d\eta \quad \text{for } -\sigma \leqslant y \leqslant \sigma.$$