J. Math. Anal. Appl. 411 (2014) 732–741

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weighted Fourier–Laplace transforms in reproducing kernel Hilbert spaces on the sphere

T. Jordão, V.A. Menegatto

ICMC-USP – São Carlos, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil

ARTICLE INFO

Article history:
Received 25 January 2012
Available online 14 October 2013
Submitted by D. Waterman

Keywords:
Sphere
Reproducing kernel Hilbert spaces
Fourier–Laplace transforms
Fractional derivative
Laplace–Beltrami derivative

ABSTRACT

We study the action of a weighted Fourier–Laplace transform on the functions in the reproducing kernel Hilbert space (RKHS) associated with a positive definite kernel on the sphere. After defining a notion of smoothness implied by the transform, we show that smoothness of the kernel implies the same smoothness for the generating elements (spherical harmonics) in the Mercer expansion of the kernel. We prove a reproducing property for the weighted Fourier–Laplace transform of the functions in the RKHS and embed the RKHS into spaces of smooth functions. Some relevant properties of the embedding are considered, including compactness and boundedness. The approach taken in the paper includes two important notions of differentiability characterized by weighted Fourier–Laplace transforms: fractional derivatives and Laplace–Beltrami derivatives.

1. Introduction

Reproducing kernel Hilbert spaces (RKHS) enter in the formulation of many problems in Approximation Theory, Learning Theory, Signal Analysis, Geomathematics, etc. They comprise an interesting and relevant class of function spaces in these areas of mathematics and many others as well. In addition to the metric structure they carry, their so-called reproducing property is an important tool in many theoretical aspects, prevailing in a decisive manner in the implementation of many procedures in the problems where they appear. We mention references [3,5,6,8,20,21] and others quoted there for a general discussion on RKHS, including potential applications.

In this paper the focus will be on RKHS based on the unit sphere S^m of $\mathbb{R}^{m+1}$, $m \geqslant 2$, endowed with its usual surface element $d\sigma_m$. The RKHS will be constructed from a Mercer kernel on S^m, that is, a function $K: S^m \times S^m \to \mathbb{C}$ possessing a series representation in the form

$$K(x,y) = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k,m)} \lambda_{k,l} Y_{k,l}(x) \overline{Y_{k,l}(y)}, \quad x, y \in S^m, \tag{1.1}$$

in which

$$\{Y_{k,l}\colon l = 1, 2, \dots, N(k,m)\}$$

is an orthonormal basis of the space $\mathcal{H}_k^m$ of spherical harmonics of degree k in $m+1$ variables and $\lambda_{k,l} > 0$, $l = 1, 2, \dots, N(k,m)$, $k = 0, 1, \dots$. The summability assumption

E-mail addresses: thsjordao@gmail.com (T. Jordão), menegatt@icmc.usp.br (V.A. Menegatto).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.10.020

$$\sum_{k=0}^{\infty} k^{m-1} \sum_{l=1}^{N(k,m)} \lambda_{k,l} < \infty, \tag{1.2}$$

as suggested in [19, p. 360], is an easy way to guarantee that K is finitely valued. That can be ratified via the Cauchy–Schwarz inequality plus a basic estimate for spherical harmonics [17, p. 28].

The RKHS associated to K, here denoted by $(\mathcal{H}_K, \langle\cdot,\cdot\rangle_K)$, or simply $\mathcal{H}_K$, is the unique Hilbert space possessing the following features:

(i) $K^x := K(\cdot, x) \in \mathcal{H}_K$, $x \in X$;
(ii) $f(x) = \langle f, K^x\rangle_K$, $f \in \mathcal{H}_K$, $x \in X$ (reproducing property).

In an earlier work [12], assuming certain differentiability assumptions on the kernel K, we deduced differentiability properties for the functions in $\mathcal{H}_K$. There are many ways to consider differentiability of functions with domain S^m and the option chosen in [12] was the usual differentiability as described in [10,17,18] and other references as well. Among many things proved there we mention the fact that, in a certain sense, functions in $\mathcal{H}_K$ inherit the same order of differentiability of the generating kernel K. Other references discussing usual differentiability in RKHS are [7,26,30].

In the present paper the mainstream will not change but the basic idea is to consider a framework that includes fractional derivatives. Hence, we will consider an operator defined by a convenient weighted Fourier–Laplace series and will analyze the action of such an operator on the elements of the RKHS. For a quick account on a particular case of such transforms, namely, the fractional derivatives, we refer the reader to [28]; the paper [1] contains information regarding the origin and motivation for such a concept. Among the main features of our approach is the fact that it also includes the so-called Laplace–Beltrami derivative as surveyed in [16], a concept that has popped up quite frequently in many problems in analysis on the sphere (see [4] for example).

The actual goals intended in this paper along with its layout are described below. In Section 2 we introduce notation, revise upon Fourier–Laplace series expansions and introduce the spaces of smooth functions implied by the operators which are pertinent to this work. In Section 3, we assume smoothness of the kernel with respect to one of the variables and show that all the functions in $\mathcal{H}_K$ inherit the same smoothness. In particular, we compute formulas for the action of the operator on the elements of $\mathcal{H}_K$. Section 4 describes some properties of the embedding of $\mathcal{H}_K$ into spaces of functions which are smooth in the sense considered in the paper.

The results to be deduced, despite producing theoretical conclusions, are motivated by problems from Learning Theory where many questions similar to the ones to be considered here were investigated using different notions of differentiability. Some references quoted at the end of this paper ratify that and also the applicability of results of this nature. For instance, two examples involving gradients (and therefore, derivatives) are presented in [30]: semi-supervised learning with gradients of functions in the RKHS and Hermite learning with gradient data. As for the embedding of the RKHS into spaces of smooth functions, they are usually attached to problems of the existence of covering numbers and their use in the estimation of approximation errors [2,22,23,30,29]. The differentiability of the functions in the RKHS enters in the algorithms and in the estimation of the covering numbers themselves. To our knowledge, the setting to be considered here, that is, the basic space being a sphere and the derivative being given by transforms acting on kernel expansions was not explored before.

2. Basic concepts and setting

Hereafter, by a Mercer kernel we will mean a kernel fitting into the description given in (1.1). Let us write $\|\cdot\|_p$ $(p \geqslant 2)$ to denote the usual norm in $L^p(S^m) := L^p(S^m, \sigma_m)$. The corresponding inner product from the case $p = 2$ will be written as $\langle\cdot,\cdot\rangle_2$.

If a Mercer kernel K defines an element in the space $L^p(S^m \times S^m) := L^p(S^m \times S^m, \sigma_m \times \sigma_m)$, the formula

$$L_K(f) = \int_{S^m} K(\cdot, y) f(y)\, d\sigma_m(y) \tag{2.1}$$

defines the integral operator $L_K : L^p(S^m) \to L^p(S^m)$ generated by K. The boundedness of L_K follows from Hölder's inequality and the fact that $\sigma_m(S^m) < \infty$ while its compactness follows from general results in Functional Analysis [13, p. 326]. The formula

$$L_K(Y_{k,l}) = \lambda_{k,l} Y_{k,l}, \quad l = 1, 2, \ldots, N(k,m),\ k = 0, 1, \ldots \tag{2.2}$$

implies the well-known property $Y_{k,l} \in L^p(S^m)$, $l = 1, 2, \ldots, N(k,m)$, $k = 0, 1, \ldots.$

Several arguments in the paper will require the inclusion $\mathcal{H}_K \subset L^p(S^m)$. The additional assumption we will need in order to guarantee this inclusion involves the mapping $\kappa_0 : S^m \to \mathbb{R}$ given by the formula $\kappa_0(x) = K(x,x)$, $x \in S^m$. The positive definiteness of K implies the inequality

$$\left|K(x,y)\right|^2 \leqslant \kappa_0(x)\kappa_0(y), \quad x, y \in S^m,$$