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Iteration of self-maps on a product of Hilbert balls

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ABSTRACT

Let $D = D_1 \times \cdots \times D_p$ be a product of Hilbert balls, with coordinate maps $\pi_j : \overline{D} \to \overline{D}_j$ on the closure $\overline{D}$, for $j = 1, \ldots, p$. Let f be a fixed-point free self-map on D, which is nonexpansive in the Kobayashi distance, and compact for $p \geqslant 2$. We describe the horospheres invariant under f and show that there exist a boundary point $(\xi_1, \ldots, \xi_p)$ of D and a nonempty set $J \subset \{1, \ldots, p\}$ such that each limit function h of the iterates (f^n) satisfies $\xi_j \in \overline{\pi_j \circ h(D)}$ for all $j \in J$ and $\pi_j \circ h(\cdot) = \xi_j$ whenever $\pi_j \circ h(D)$ meets the boundary of D_j. For a single Hilbert ball D_1, either $\liminf_{n\to\infty} \|f^{2n}(0)\| < 1$ or (f^n) converges locally uniformly to a constant map taking value at the boundary of D_1.

1. Introduction

Iteration of holomorphic maps on a Hilbert ball has been widely studied (see, for example, [2,3,6,10,12,17,18,20,22]). In infinite dimension, several authors [6,13,17] have studied iteration of compact holomorphic maps to which most finite dimensional results, including the Denjoy–Wolff theorem for the unit disc, can be extended. In this paper, we study iteration of a self-map f on a cartesian product $D = D_1 \times \cdots \times D_p$ of Hilbert balls, which is nonexpansive in the Kobayashi distance, and compact if $p \geqslant 2$. If f does not have a fixed point in D, we first give an explicit description in Proposition 2.4 of the f-invariant horospheres at a boundary point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$ of D. We then use this to show in Theorem 3.2 that there is a nonempty set $J \subset \{1, \ldots, p\}$ such that each *limit function* h of the iterates (f^n) satisfies $\xi_j \in \overline{\pi_j \circ h(D)}$ for all $j \in J$ and $\pi_j \circ h(\cdot) = \xi_j$ whenever $\pi_j \circ h(D)$ meets the boundary of D_j, where π_j is the coordinate map $(x_1, \ldots, x_p) \in \overline{D} \mapsto x_j \in \overline{D}_j$, and a limit function h is a subsequential limit of (f^n) (cf. Definition 3.1). This generalises the results in [1,11] for the polydiscs and unifies the iteration results for a single Hilbert ball. Indeed, if $D = D_1$ is a single Hilbert ball, then the result states that $h(\cdot) = \xi_1$ for each limit function h of (f^n) having a value in the boundary of D_1. In general, h can have values inside D_1 even if f is biholomorphic (cf. [22]). However, if D_1 is finite dimensional or if f is compact, then all limit functions of (f^n) have values in the boundary of D_1 and hence they are identical to the constant map $h(\cdot) = \xi_1$. It follows that (f^n) converges locally uniformly to h, which is the generalised Denjoy–Wolff theorem proved in [6,12,17,18]. Without compactness condition on f, the Denjoy–Wolff theorem may fail for infinite dimensional Hilbert balls. Nevertheless, compactness is not a necessary condition for the Denjoy–Wolff theorem to hold. For a self-map f, nonexpansive in the Kobayashi distance on a single Hilbert ball D_1, we give necessary and sufficient conditions in Corollary 3.6 for the Denjoy–Wolff theorem to hold. In particular, we show that either $\liminf_{n\to\infty} \|f^{2n}(0)\| < 1$ or (f^n) converges locally uniformly to a constant map $h(\cdot) = \xi_1$ with $\|\xi_1\| = 1$. This improves and simplifies the results in [6,17] for compact maps, as well as giving some perspective of the failure of the Denjoy–Wolff theorem in the example of [22]. We also show that, if a fixed-point free self-map f contracts the Kobayashi distance in D_1, then the existence of a totally bounded orbit $(f^n(x_1))$ is sufficient (and necessary) for the Denjoy–Wolff theorem to hold.

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Besides the Kobayashi metric, we make use of the Jordan algebraic structure of the ambient space of D, which is a JB*-triple. Especially, we use the Bergmann operator to describe the invariant horospheres, alternative to the ones given in [1] for polydiscs and other domains, but well suited for D and providing a generalised version of Wolff's theorem for the unit disc in $\mathbb{C}$. To prove Theorem 3.2, we determine the intersection of a horosphere with the boundary of D which, unlike the case of a single Hilbert ball, need not be a singleton.

In what follows, $D = D_1 \times \cdots \times D_p$ always denotes the cartesian product of a finite number of open unit balls $D_j = \{v \in V_j \colon \|v\| < 1\}$ in complex Hilbert spaces V_j, with inner product $\langle \cdot,\cdot\rangle_j$, for $j = 1, \dots, p$. We observe that D is the open unit ball of the ℓ^∞-sum V of the Hilbert spaces $V_1, \dots, V_p$:

$$V = V_1 \oplus_\infty \cdots \oplus_\infty V_p$$

where $\|(x_1,\dots,x_p)\| = \sup\{\|x_1\|,\dots,\|x_p\|\}$ for $(x_1,\dots,x_p) \in V$. We shall call D_j a *Hilbert ball*, and D a *polyball* in the sequel. If each D_j is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ in the complex plane, D is usually called a polydisc.

There is a Jordan triple structure in each Hilbert space V_j, derived from the geometry of D_j. Indeed, V_j is equipped with a Jordan triple product

$$\{\cdot,\cdot,\cdot\}_j : V_j^3 \to V_j$$

defined by

$$\{a,b,c\}_j = \frac{1}{2}\big(\langle a,b\rangle_j c + \langle c,b\rangle_j a\big) \quad (a,b,c \in V_j).$$

With this triple product, V_j becomes a JB*-triple. The ℓ^∞-sum V is also a JB*-triple in the coordinatewise Jordan triple product:

$$\big\{(x_1,\dots,x_p),(y_1,\dots,y_p),(z_1,\dots,z_p)\big\} = \big(\{x_1,y_1,z_1\}_1,\dots,\{x_p,y_p,z_p\}_p\big)$$

for $(x_1,\dots,x_p),(y_1,\dots,y_p),(z_1,\dots,z_p) \in V$. The subscript j for the inner product and triple product are often omitted if no confusion is likely.

We refer to the seminal paper [14] for the origin of JB*-triples and to [4,7,23] for the basics of JB*-triples as well as proofs of some facts used below. The relevance of JB*-triples rests with the fact that bounded symmetric domains in complex Banach spaces are biholomorphically the open unit balls of JB*-triples [14].

The biholomorphic maps of the open unit ball U of a JB*-triple Z with Jordan triple product $\{\cdot,\cdot,\cdot\}$ are determined by the Möbius transformations. The latter can be described as follows.

Given $a \in U$ and $b, c \in Z$, the *box operator* $b \,\square\, c : Z \to Z$ and the *Bergmann operator* $B(b,c) : Z \to Z$ are continuous linear maps defined by

$$(b \,\square\, c)(x) = \{b,c,x\},$$
$$B(b,c)(x) = x - 2(b \,\square\, c)(x) + \big\{b,\{c,x,c\},b\big\} \quad (x \in Z).$$

The *Möbius transformation* $g_a : U \to U$, induced by a, is given by

$$g_a(z) = a + B(a,a)^{1/2}(I + z \,\square\, a)^{-1}(z) \quad (z \in U)$$

where I is the identity operator, the Bergmann operator $B(a,a)$ has positive spectrum and the operator $I + z \,\square\, a$ is invertible for $z, a \in U$. The inverse of g_a is the map g_{-a} and g_a has no fixed point in U unless $a = 0$.

The *Kobayashi distance* κ on U can be expressed in terms of the Möbius transformation:

$$\kappa(a,b) = \tanh^{-1}\big\|g_{-b}(a)\big\| \quad (a,b \in U)$$

(cf. [9, p. 85]), where we have

$$1 - \big\|g_{-b}(a)\big\|^2 = \big\|B(a,a)^{-1/2}B(a,b)B(b,b)^{-1/2}\big\|^{-1} \tag{1.1}$$

as shown in [19] (see also [4]). We will often make use of the identity

$$\big\|B(a,a)^{-1/2}\big\| = \frac{1}{1 - \|a\|^2} \quad (a \in U) \tag{1.2}$$

(cf. [4, p. 194]).

For the open unit ball D_j of a Hilbert space V_j and $a \in D_j$, the Bergmann operator $B(a,a)$ is a self-adjoint operator on V_j and its square root is given by

$$B(a,a)^{1/2}(z) = \sqrt{1 - \|a\|^2}\left(z + \big(\sqrt{1 - \|a\|^2} - 1\big)\langle z,a\rangle_j \frac{a}{\|a\|^2}\right) \tag{1.3}$$