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Orbital and strongly orbital spaces

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ABSTRACT

We say that a (countably dimensional) topological vector space X is orbital if there is $T \in L(X)$ and a vector $x \in X$ such that X is the linear span of the orbit $\{T^n x : n = 0, 1, \dots\}$. We say that X is strongly orbital if, additionally, x can be chosen to be a hypercyclic vector for T. Of course, X can be orbital only if the algebraic dimension of X is finite or infinite countable. We characterize orbital and strongly orbital metrizable locally convex spaces. We also show that every countably dimensional metrizable locally convex space X does not have the invariant subset property. That is, there is $T \in L(X)$ such that every non-zero $x \in X$ is a hypercyclic vector for T. Finally, assuming the Continuum Hypothesis, we construct a complete strongly orbital locally convex space.

As a byproduct of our constructions, we determine the number of isomorphism classes in the set of dense countably dimensional subspaces of any given separable infinite dimensional Fréchet space X. For instance, in $X = \ell_2 \times \omega$, there are exactly 3 pairwise non-isomorphic (as topological vector spaces) dense countably dimensional subspaces.

1. Introduction

All vector spaces in this article are over the field $\mathbb{K}$ being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Throughout the article, all topological spaces *are assumed to be Hausdorff*. Recall that a *Fréchet space* is a complete metrizable locally convex space. For a topological vector space X, $L(X)$ is the algebra of continuous linear operators on X and X' is the space of continuous linear functionals on X. Symbol $GL(X)$ stands for the group of invertible $T \in L(X)$ for which $T^{-1} \in L(X)$. For $T \in L(X)$, the dual operator $T' : X' \to X'$ is defined as usual: $T'f = f \circ T$.

By saying *countable*, we always mean infinite countable. Recall that the topology τ of a topological vector space X is called *weak* if τ is the weakest topology making each $f \in Y$ continuous for some linear space Y of linear functionals on X separating points of X. It is well-known and easy to see that a topology of a metrizable infinite dimensional topological vector space X is weak if and only if X is isomorphic to a dense linear subspace of $\omega = \mathbb{K}^{\mathbb{N}}$. This observation makes ω the only infinite dimensional Fréchet space, whose topology is weak, see [11,25] for more details. Another property of ω we shall use on a number of occasions, is that if a subspace Y of a locally convex topological vector space X is isomorphic to ω, then Y is complemented, see, for instance, [11] for the proof. For more information on the theory of Fréchet spaces see the survey [9] and references therein.

Recall that a topological vector space X has the *invariant subspace property* if every $T \in L(X)$ has a non-trivial (= different from $\{0\}$ and X) closed invariant subspace. Similarly, a topological vector space X has the *invariant subset property* if every $T \in L(X)$ has a non-trivial (= different from $\varnothing$, $\{0\}$ and X) closed invariant subset. The problem of whether ℓ_2 has the invariant subspace property is the famous invariant subspace problem. It is worth noting that Read [21] and Enflo [15] (see also [5]) showed independently that there are separable infinite dimensional Banach spaces, which do not have the invariant subspace property. In fact, Read [22,23] demonstrated that ℓ_1 does not have the invariant subset property. It is worth noting

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that Atzmon [2,3] constructed an infinite dimensional nuclear Fréchet space without the invariant subspace property. All existing constructions of operators on Banach spaces with no invariant subspaces are rather sophisticated. On the other hand, examples of separable non-complete normed spaces with or without the invariant subspace or subset property are relatively easy to construct.

Recall that $x \in X$ is called a *hypercyclic vector* for $T \in L(X)$ if the orbit

$$O(T, x) = \{T^n x : n \in \mathbb{Z}_+\}$$

is dense in X. Similarly, x is called *cyclic* for T if the linear span of $O(T, x)$ is dense in X. An operator T is called *hypercyclic* (respectively, *cyclic*) if it possesses a hypercyclic (respectively, cyclic) vector. For more information on hypercyclicity see the books [4,18] and references therein. Clearly, T has no non-trivial invariant subsets (respectively, subspaces) precisely when every non-zero vector is hypercyclic (respectively, cyclic) for T. If the space in question is countably dimensional, there is an unusual way to approach the invariant subspace/subset property.

Definition 1.1. We say that a topological vector space X is *orbital* if there exist $T \in L(X)$ and $x \in X$ such that $X = \mathtt{span}(O(T, x))$. We say that X is *strongly orbital* if there exist $T \in L(X)$ and $x \in X$ such that x is a hypercyclic vector for T and $X = \mathtt{span}(O(T, x))$.

Note that an orbital space always has either finite or countable dimension. Since finite dimensional topological vector spaces support no hypercyclic operators, every strongly orbital space is countably dimensional. We start with the following easy observation.

Lemma 1.2. *Each strongly orbital topological vector space X does not have the invariant subset property.*

Proof. Let X be a topological vector space and $T \in L(X)$ and $x \in X$ be such that x is a hypercyclic vector for T and $X = \mathtt{span}(O(T, x))$. Due to Wengenroth [29], $p(T)(X)$ is dense in X for every non-zero polynomial p. It suffices to show that T has no non-trivial invariant subsets. Take $y \in X \setminus \{0\}$. Since $X = \mathtt{span}(O(T, x))$, there is a non-zero polynomial p such that $y = p(T)x$. Hence

$$O(T, y) = O(T, p(T)y) = p(T)(O(T, x))$$

and therefore $O(T, y)$ is dense in X since each $p(T)(X)$ is dense in X and $O(T, x)$ is dense in X. Thus every non-zero vector is hypercyclic for T, which means that T has no non-trivial invariant subsets. □

The following two propositions are known facts and are basically a compilation of certain results from [10,17]. We present their proofs for the sake of convenience.

Proposition 1.3. *Every normed space X of countable algebraic dimension is strongly orbital and therefore does not have the invariant subset property.*

Proof. Let B be the completion of X. Then B is a separable infinite dimensional Banach space. According to Ansari [1] and Bernal-Gonzáles [6], there is a hypercyclic $T \in L(B)$. That is, there is $x \in B$ such that $\{T^n x : n \in \mathbb{Z}_+\}$ is dense in B. Let Z be the linear span of $O(T, x)$. Since both Z and X are dense countably dimensional linear subspaces of B, according to Grivaux [17], there is $S \in GL(B)$ such that $S(Z) = X$. Since Z is invariant for T, X is invariant for STS^{-1}. Hence, the restriction $A = STS^{-1}|_X$ belongs to $L(X)$, $O(A, Sx) = S(O(T, x))$ is dense and $X = \mathtt{span}(O(A, Sx))$. Thus X is strongly orbital. By Lemma 1.2, X does not have the invariant subset property. □

Proposition 1.4. *In every separable infinite dimensional Fréchet space X, there is a dense linear subspace E such that E has the invariant subspace property.*

Proof. There is a dense linear subspace E of X (E can even be chosen to be a hyperplane [11]) such that every $T \in L(E)$ has the shape $\lambda I + S$ with $\lambda \in \mathbb{K}$ and $\dim S(X) < \infty$. Trivially, such a T has a one-dimensional invariant subspace. □

1.1. Results

We partially extend Proposition 1.3 from the class of normed spaces to the class of metrizable locally convex (topological vector) spaces.

Theorem 1.5. *Every metrizable locally convex space X of countable algebraic dimension does not have the invariant subset property.*

It turns out that not every metrizable locally convex space X of countable algebraic dimension is strongly orbital or even orbital. In order to formulate the result neatly, we split the class $\mathcal{M}$ of infinite dimensional metrizable locally convex spaces into 4 subclasses. Given $X \in \mathcal{M}$ and an increasing sequence $\{p_n\}_{n\in\mathbb{N}}$ of seminorms on X defining the topology of X, we have the following 4 possibilities:

(A_0) $X/\mathtt{ker}\, p_1$ is finite dimensional and $\mathtt{ker}\, p_n/\mathtt{ker}\, p_{n+1}$ is finite dimensional for each $n \in \mathbb{N}$;
(A_1) there is $n \in \mathbb{N}$ such that $\mathtt{ker}\, p_n = \{0\}$;