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Integral manifolds for partial functional differential equations in admissible spaces on a half-line ☆

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ABSTRACT

In this paper we investigate the existence of stable and center-stable manifolds for solutions to partial functional differential equations of the form $\dot{u}(t) = A(t)u(t) + f(t, u_t)$, $t \geqslant 0$, when its linear part, the family of operators $(A(t))_{t\geqslant 0}$, generates the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ having an exponential dichotomy or trichotomy on the half-line and the nonlinear forcing term f satisfies the φ-Lipschitz condition, i.e., $\|f(t,u_t) - f(t,v_t)\| \leqslant \varphi(t)\|u_t - v_t\|_{\mathcal{C}}$ where $u_t, v_t \in \mathcal{C} := C([-r,0],X)$, and $\varphi(t)$ belongs to some admissible function space on the half-line. Our main methods invoke Lyapunov–Perron methods and the use of admissible function spaces.

1. Introduction

Consider the partial functional differential equation

$$\frac{du}{dt} = A(t)u(t) + f(t, u_t), \quad t \in [0, +\infty), \tag{1.1}$$

where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed t; $f : \mathbb{R}_+ \times \mathcal{C} \to X$ is a continuous nonlinear operator with $\mathcal{C} := C([-r,0],X)$, and u_t is the history function defined by $u_t(\theta) := u(t+\theta)$ for $\theta \in [-r,0]$. When the family of operators $(A(t))_{t\geqslant 0}$ generates the evolution family having an exponential dichotomy (or trichotomy), one tries to find conditions on the nonlinear forcing term f such that Eq. (1.1) has an integral manifold (e.g., a stable, unstable or center manifold). The most popular condition imposed on f is its uniform Lipchitz continuity with a sufficiently small Lipschitz constant, i.e., $\|f(t,\phi) - f(t,\psi)\| \leqslant q\|\phi - \psi\|_{\mathcal{C}}$ for q small enough (see [1,3,14] and references therein). However, for equations arising in complicated reaction–diffusion processes, the function f represents the source of material (or population) which, in many contexts, depends on time in diversified manners (see [5, Chapt. 11], [6,15]). Therefore, sometimes one may not hope to have the uniform Lipschitz continuity of f. Recently, for the case of partial differential equations *without delay*, we have obtained exciting results in [9], where we have used the Lyapunov–Perron method and the characterization of the exponential dichotomy (obtained in [8]) of evolution equations in admissible function spaces to construct the structures of solutions in mild forms, which belong to some certain classes of admissible spaces on which we could employ some well-known principles in mathematical analysis such as the contraction mapping principle, the implicit function theorem, etc. The use of admissible spaces has helped us to construct the invariant manifolds without using the smallness of Lipschitz constants of nonlinear forcing terms in classical sense (see [9,10]).

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Another point we would like to mention is that in some applications the partial differential operator $A(t)$ is defined only for $t \geqslant 0$ (see e.g., [4,11,13] and references therein). Therefore, the evolution family generated by $(A(t))_{t\geqslant 0}$ is defined only on a *half-line*.

The purpose of the present paper is to prove the existence of stable and center-stable manifolds for Eq. (1.1) when its linear part $(A(t))_{t\geqslant 0}$ generates the evolution family having an exponential dichotomy or trichotomy on the *half-line* under more general conditions on the nonlinear forcing term f, that is the φ-Lipschitz continuity of f, i.e., $\|f(t,\phi) - f(t,\psi)\| \leqslant \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$ where $\phi, \psi \in \mathcal{C}$, and $\varphi(t)$ is a real and positive function which belongs to admissible function spaces defined in Definition 2.3 below. We will extend the methods in [9] to the case of partial functional differential equations (PFDE). The main difficulties that we face when passing to the case of PFDE are the following two features: Firstly, since the nonlinear operator f is φ-Lipschitz, the existence and uniqueness theorem for solutions to (1.1) is not available. Secondly, the evolution family generated by $(A(t))_{t\geqslant 0}$ is defined only on a half-line $\mathbb{R}_+$ and doesn't act on the same Banach space as that the surfaces of the integral manifold belong to (in fact, the former acts on X, and the latter belongs to $\mathcal{C}$). Therefore, the standard methods of nonlinear perturbations of an evolutionary process using graph transforms as formulated in [1,3] cannot be applied here.

To overcome such difficulties, we reformulate the definition of invariant manifolds such that it contains the existence and uniqueness theorem as a property of the manifold (see Definition 3.3 below). Furthermore, we construct the structure of the mild solutions to (1.1) using the Lyapunov–Perron equation (see Eq. (3.5)) in such a way that it allows to combine the exponential dichotomy of the linear part of Eq. (1.1) with the existence and uniqueness of its bounded solutions in the case of φ-Lipschitz nonlinear forcing terms. Then, we use the admissible spaces to construct the integral manifolds for Eq. (1.1) in the case of dichotomic linear part without using the smallness of Lipschitz constants of the nonlinear terms in classical sense. Instead, the "smallness" is now understood as the sufficient smallness of $\sup_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)\,d\tau$. Consequently, we obtain the existence of invariant stable manifolds for the case of dichotomic linear parts under very general conditions on the nonlinear term $f(t,u_t)$. Moreover, using these results and rescaling procedures we prove the existence of center-stable manifolds for the mild solutions to Eq. (1.1) in the case of trichotomic linear parts under the same conditions on the nonlinear term f as in the dichotomic case. Our main results are contained in Theorems 3.7, 4.2.

We now recall some notions.

For a Banach space X (with a norm $\|\cdot\|$) and a given $r > 0$ we denote by $\mathcal{C} := C([-r,0],X)$ the Banach space of all continuous functions from $[-r,0]$ into X, equipped with the norm $\|\phi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]}\|\phi(\theta)\|$ for $\phi \in \mathcal{C}$. For a continuous function $v : [-r,\infty) \to X$ the *history function* $v_t \in \mathcal{C}$ is defined by $v_t(\theta) = v(t+\theta)$ for all $\theta \in [-r,0]$.

Definition 1.1. A family of bounded linear operators $\{U(t,s)\}_{t\geqslant s\geqslant 0}$ on a Banach space X is a (*strongly continuous, exponential bounded*) *evolution family* if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \geqslant r \geqslant s \geqslant 0$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
(iii) there are constants $K, c \geqslant 0$ such that $\|U(t,s)x\| \leqslant Ke^{c(t-s)}\|x\|$ for all $t \geqslant s \geqslant 0$ and $x \in X$.

The notion of an evolution family arises naturally from the theory of non-autonomous evolution equations which are well-posed. Meanwhile, if the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t), \quad t \geqslant s \geqslant 0, \\ u(s) = x_s \in X \end{cases} \tag{1.2}$$

is well-posed, there exists an evolution family $\{U(t,s)\}_{t\geqslant s\geqslant 0}$ such that the solution of the Cauchy problem (1.2) is given by $u(t) = U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [11] (see also Nagel and Nickel [7] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$).

2. Function spaces and admissibility

We recall some notions on function spaces and refer to Massera and Schäffer [2], Räbiger and Schnaubelt [12] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. The space $L_{1,loc}(\mathbb{R}_+)$ of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n}|f(t)|\,dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$ (see [2, Chapt. 2, §20]).

We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called *a Banach function space* (over $(\mathbb{R}_+,\mathcal{B},\lambda)$) if