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Generalized elementary functions

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ABSTRACT

We use the theory of generalized linear differential equations to introduce new definitions of the exponential, hyperbolic and trigonometric functions. We derive some basic properties of these generalized functions, and show that the time scale elementary functions with Lebesgue integrable arguments represent a special case of our definitions.

1. Introduction

There are many equivalent ways of introducing the classical exponential function; one possibility is to define the exponential as the unique solution of the initial-value problem $z'(t) = z(t)$, $z(0) = 1$. More generally, for every continuous function p defined on the real line, the initial-value problem $z'(t) = p(t)z(t)$, $z(t_0) = 1$, which can be written in the equivalent integral form

$$z(t) = 1 + \int_{t_0}^{t} p(s)z(s)\,\mathrm{d}s, \tag{1.1}$$

has the unique solution $z(t) = e^{\int_{t_0}^{t} p(s)\,\mathrm{d}s}$. In this paper, we replace Eq. (1.1) by the more general equation

$$z(t) = 1 + \int_{t_0}^{t} z(s)\,\mathrm{d}P(s), \tag{1.2}$$

where the integral on the right-hand side is the Kurzweil–Stieltjes integral (see the next section). We define the generalized exponential function $e_{\mathrm{d}P}$ as the unique solution of Eq. (1.2) and study its properties. Obviously, if P is continuously differentiable with $P' = p$, then Eq. (1.2) reduces back to Eq. (1.1). On the other hand, Eq. (1.2) is much more general and makes sense even if P is discontinuous. We point out that Eq. (1.2) is a generalized linear differential equation in the sense of J. Kurzweil's definition [10]. Therefore, we can use the existing theory of generalized differential equations (see [13,14]) in our study of the generalized exponential function.

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In Section 2, we start by recalling some basic facts about the Kurzweil–Stieltjes integral and generalized linear ordinary differential equations. Next, we prove an existence-uniqueness theorem for equations with complex-valued coefficients and solutions. In Section 3, we define the generalized exponential function and investigate its properties. For example, we show that the product of two exponentials $e_{dP}e_{dQ}$ gives the exponential of a function denoted by $P \oplus Q$; when P, Q are continuous, we have $P \oplus Q = P + Q$, but otherwise $P \oplus Q$ is more complicated and takes into account the discontinuities of P and Q. In Section 4, we use the exponential to introduce the generalized hyperbolic and trigonometric functions. Finally, in Section 5, we demonstrate that our generalized elementary functions include the time scale elementary functions as a special case. At the same time, we show how the usual definitions of the time scale exponential, hyperbolic and trigonometric functions can be extended from rd-continuous to Lebesgue Δ-integrable arguments.

2. Preliminaries

We need the concept of the Kurzweil–Stieltjes integral, which represents a special case of the Kurzweil integral introduced in [10] under the name “generalized Perron integral”.

Consider a pair of functions $g : [a,b] \to \mathbb{R}^{n\times n}$ and $f : [a,b] \to \mathbb{R}^n$. We say that f is Kurzweil–Stieltjes integrable with respect to g, if there exists a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a function $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{j=1}^{m} \big(g(s_j) - g(s_{j-1})\big) f(\tau_j) - I \right\| < \varepsilon \tag{2.1}$$

for every partition of $[a,b]$ with division points $a = s_0 < s_1 < \cdots < s_m = b$ and tags $\tau_j \in [s_{j-1}, s_j]$, $j \in \{1,\ldots,m\}$, satisfying

$$[s_{j-1}, s_j] \subset \big(\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)\big), \quad j \in \{1,2,\ldots,m\}.$$

In this case, the vector I is called the Kurzweil–Stieltjes integral (or the Perron–Stieltjes integral), and we use the notation $\int_a^b \mathrm{d}[g]\, f = I$.

In this paper, the case when $n > 1$ is needed only in the formulation of Theorem 2.6, which is then used to prove Theorem 2.7. For $n = 1$, the order of multiplication in (2.1) does not matter, i.e.,

$$\sum_{j=1}^{m} \big(g(s_j) - g(s_{j-1})\big) f(\tau_j) = \sum_{j=1}^{m} f(\tau_j)\big(g(s_j) - g(s_{j-1})\big),$$

and we occasionally use the simpler notation $\int_a^b f\, \mathrm{d}g$ in place of $\int_a^b \mathrm{d}[g]\, f$.

Basic properties of the Kurzweil–Stieltjes integral can be found in [13,17].

Throughout this paper, we work with regulated functions defined on a compact interval $[a,b]$ and use the following notation:

$$\Delta^+ g(t) = \begin{cases} g(t+) - g(t) & \text{if } t \in [a,b), \\ 0 & \text{if } t = b, \end{cases} \qquad \Delta^- g(t) = \begin{cases} g(t) - g(t-) & \text{if } t \in (a,b], \\ 0 & \text{if } t = a. \end{cases}$$

Also, we let $\Delta g(t) = \Delta^+ g(t) + \Delta^- g(t)$.

The following theorem describes the properties of the indefinite Kurzweil–Stieltjes integral and can be found in [17, Proposition 2.16].

Theorem 2.1. *Consider a pair of functions $f, g : [a,b] \to \mathbb{R}$ such that g is regulated and $\int_a^b f\, \mathrm{d}g$ exists. Then, for every $t_0 \in [a,b]$, the function*

$$h(t) = \int_{t_0}^{t} f\, \mathrm{d}g, \quad t \in [a,b]$$

is regulated and satisfies

$$h(t+) = h(t) + f(t)\Delta^+ g(t), \quad t \in [a,b),$$
$$h(t-) = h(t) - f(t)\Delta^- g(t), \quad t \in (a,b].$$

If $I \subset \mathbb{R}$ is an interval, $h : I \to \mathbb{R}$ is a function which is zero except a countable set $\{t_1, t_2, \ldots\} \subset I$, and the sum $S = \sum_i h(t_i)$ is absolutely convergent, we use the notation $S = \sum_{x \in I} h(x)$. The next lemma is taken over from [17, Proposition 2.12].