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On some relative convexities

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ABSTRACT

We study two types of relative convexities of convex functions f and g. We say that f is convex relative to g in the sense of Palmer (2002, 2003), if $f = h(g)$, where h is strictly increasing and convex, and denote it by $f \succ_{(1)} g$. Similarly, if f is convex relative to g in the sense studied in Rajba (2011), that is if the function $f - g$ is convex then we denote it by $f \succ_{(2)} g$. The relative convexity relation $\succ_{(2)}$ of a function f with respect to the function $g(x) = cx^2$ means the strong convexity of f. We analyze the relationships between these two types of relative convexities. We characterize them in terms of right derivatives of functions f and g, as well as in terms of distributional derivatives, without any additional assumptions of twice differentiability. We also obtain some probabilistic characterizations. We give a generalization of strong convexity of functions and obtain some Jensen-type inequalities.

1. Introduction

Let $I = (a, b)$ $(-\infty \leqslant a < b \leqslant \infty)$ be an open interval. Recall that the function $f : I \to \mathbb{R}$ is *convex*, if the inequality

$$f\big(tx + (1-t)y\big) \leqslant tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and for all $t \in [0, 1]$ (see [11,24]). Convexity has a nice characterization. A function $f : I \to \mathbb{R}$ is convex if and only if

$$f(EX) \leqslant Ef(X) \tag{1.2}$$

for all I-valued integrable random variables X (see [3]).

Note, that for invertible f, (1.2) can also be expressed as $EX \leqslant f^{-1}(Ef(X))$. The term $f^{-1}(Ef(X))$ can be seen as a generalized mean value [8,6]. The notion of convexity was generalized by B. Jessen in [8] to compare two functions in terms of the means defined by them (the comparative convexity). An increasing function $f : I \to \mathbb{R}$ is said to be convex with respect to another increasing function $g : I \to \mathbb{R}$ if $g^{-1}(Eg(X)) \leqslant f^{-1}(Ef(X))$. This can be written as $f \circ g^{-1}(Eg(X)) \leqslant Ef \circ g^{-1}(g(X))$, which shows that f is convex with respect to g if and only if $f \circ g^{-1}$ is convex. Note that the later formulation, used in [6,24,18], does not require f to be invertible (cf. also [4,5,2,14,15,12]). In the context of C^1-differentiable functions, f is convex with respect to an increasing g if $f'(x)/g'(x)$ is non-decreasing; in the context of C^2-differentiable functions, f is convex with respect to g if and only if $f''(x)/f'(x) \geqslant g''(x)/g'(x)$ [4] (provided these ratios exist). The relative convexity, by this definition, is not antisymmetric and thus does not define a partial ordering.

Palmer in [16,17] used an independently derived formulation that is similar, but is antisymmetric and defines a partial ordering, without requiring invertibility of either function in the relation. Following Palmer [16,17] we say that a function $f : I \to \mathbb{R}$ is convex relative to another function $g : I \to \mathbb{R}$, if there exists a strictly increasing and convex function

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$h: g(I) \to \mathbb{R}$ such that $f = h(g)$, and denote it by $f \succ_{(1)} g$. In the context of C^2-differentiable functions, $f \succ_{(1)} g$ if and only if $f''(x)/|f'(x)| \geqslant g''(x)/|g'(x)|$ $(x \in I)$, provided these ratios exist (see [17, Theorem 4]).

In [15] the authors consider a relative convexity of f with respect to g, by the terminology in [15] abbreviated $g \lhd f$, where f, g are real-valued functions defined on the same set A and g is not a constant function. When A is an interval and g is continuous and strictly increasing, the condition $g \lhd f$ is equivalent to the convexity of $f \circ g^{-1}$ (on the interval $B = g(A)$).

Another concept of relative convexity of two convex functions f and g can be found in [22]. We say that f is convex with respect to g if $f - g$ is convex, and denote it by $f \succ_{(2)} g$. In the context of C^2-differentiable functions, $f \succ_{(2)} g$ if and only if $f''(x) \geqslant g''(x)$ $(x \in I)$. In symbols we write $f \in C(\succ_{(i)}, g)$ if $f \succ_{(i)} g$ $(i = 1, 2)$. The set $C(\succ_{(2)}, cx^2)$ $(c > 0)$ coincides with the set of all strongly convex functions (with modulus $c > 0$), introduced in [20] (see also [24,7,19]).

Various other related generalizations of convexity have been proposed. Many results can be found, in [15,18,10,1,21,9], among others.

In this paper we give a characterization of the relative convexity relations $\succ_{(1)}$ and $\succ_{(2)}$ of two convex functions f and g, which generalizes and complements results of Palmer [16,17] and Rajba [22]. We analyze also the relationships between the relations $\succ_{(1)}$ and $\succ_{(2)}$.

In Section 2 we give a characterization of these relative convexities and the relationships between them, in terms of right derivatives and in terms of distributional derivatives, without any additional assumptions of twice differentiability of the functions f and g.

In Section 3 we derive a probabilistic characterization of these relative convexity relations in terms of Jensen-gaps.

In Section 4 we define and study strong convexity with respect to the relation $\succ_{(i)}$ $(i = 1, 2)$. Then the usual strong convexity can be regarded as strong convexity with respect to the relation $\succ_{(2)}$. This section also includes a probabilistic characterization of strong convexity with respect to the relation $\succ_{(1)}$, which complements the characterization of the usual strong convexity given in [23]. Using this probabilistic characterization, we derive Jensen-type inequalities for functions strongly convex with respect to $\succ_{(1)}$, analogous to the results obtained in [23,13], concerning usual strongly convex functions.

In addition to illustrating how our theorems work in practice, we provide many interesting examples and counterexamples of functions.

2. Differential criteria

As usual we denote distributional derivatives by f', pointwise derivatives by $f'(x)$, second order distributional derivatives by f'' and second order pointwise derivatives by $f''(x)$ (see [25,26]).

Recall that convex functions satisfy various smoothness properties (see [11,24]). We list some properties of a convex function.

Proposition 2.1. *A convex function f defined on I is continuous and has both right and left derivatives $f'_R(x)$ and $f'_L(x)$ at each point of I.*

Proposition 2.2. *A function $f: I \to \mathbb{R}$ is convex if and only if its right derivative $f'_R(x)$ (or left derivative $f'_L(x)$) exists and is non-decreasing on I.*

Proposition 2.3. *A function $f: I \to \mathbb{R}$ is convex if and only if its right derivative $f'_R(x)$ (or left derivative $f'_L(x)$) exists and $f'' \geqslant 0$.*

A *subderivative* of a convex function $f: I \to \mathbb{R}$ at a point $x_0 \in I$ is a real number c such that $f(x) - f(x_0) \geqslant c(x - x_0)$, for all $x \in I$. The set of all subderivatives is called *subdifferential* of a function f at x_0 and is denoted by $\partial f(x_0)$. Obviously, we have

$$\partial f(x_0) = \left[f'_L(x_0), f'_R(x_0)\right]. \tag{2.1}$$

Proposition 2.4. *A function $f: I \to \mathbb{R}$ is convex if and only if there exists a function $k: I \to \mathbb{R}$, such that for any $x, y \in I$*

$$f(y) \geqslant f(x) + k(x)(y - x). \tag{2.2}$$

Moreover, $k(x)$ is a non-decreasing function and $k(x) \in \partial f(x)$, for any $x \in I$.

By (2.1), as $k(x)$ can be taken $f'_R(x)$ (or $f'_L(x)$).

Corollary 2.5. *A function $f: I \to \mathbb{R}$ is convex if and only if*

$$f(y) \geqslant f(x) + f'_R(x)(y - x), \tag{2.3}$$

for all $x, y \in I$.