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# Operators in rigged Hilbert spaces: Some spectral properties

Giorgia Bellomonte, Salvatore Di Bella, Camillo Trapani *

*Dipartimento di Matematica e Informatica, Università di Palermo, I-90123 Palermo, Italy*

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ABSTRACT

A notion of resolvent set for an operator acting in a rigged Hilbert space $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ is proposed. This set depends on a family of intermediate locally convex spaces living between $\mathcal{D}$ and $\mathcal{D}^\times$, called interspaces. Some properties of the resolvent set and of the corresponding multivalued resolvent function are derived and some examples are discussed.



## 1. Introduction

Spaces of linear maps acting on a *rigged Hilbert space* (RHS, for short)

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$$

have often been considered in the literature both from a pure mathematical point of view [23,24,29,31] and for their applications to quantum theories (generalized eigenvalues, resonances of Schrödinger operators, quantum fields ...) [8,13,12,15,14,16,26]. The spaces of test functions and the distributions over them constitute relevant examples of rigged Hilbert spaces and operators acting on them are a fundamental tool in several problems in analysis (differential operators with singular coefficients, Fourier transforms) and also provide the basic background for the study of the problem of the multiplication of distributions by the duality method [25,28,35].

Before going forth, we fix some notations and basic definitions.

Let $\mathcal{D}$ be a dense linear subspace of Hilbert space $\mathcal{H}$ and $t$ a locally convex topology on $\mathcal{D}$, finer than the topology induced by the Hilbert norm. Then the space $\mathcal{D}^\times$ of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, is a linear vector space and *contains* $\mathcal{H}$, in the sense that $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times$. These identifications imply that the sesquilinear form $B(\cdot,\cdot)$ that puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{D}$; i.e. $B(\xi,\eta) = \langle \xi|\eta\rangle$, for every $\xi,\eta \in \mathcal{D}$ (to simplify notations we adopt the symbol $\langle\cdot|\cdot\rangle$ for both of them). The space $\mathcal{D}^\times$ will always be considered as endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$. The Hilbert space $\mathcal{H}$ is dense in $\mathcal{D}^\times[t^\times]$.

We get in this way a *Gelfand triplet* or *rigged Hilbert space* (RHS)

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times\big[t^\times\big], \tag{1}$$

where $\hookrightarrow$ denotes a continuous embedding with dense range. As it is usual, we will systematically read (1) as a chain of inclusions and we will write $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ or $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}^\times[t^\times])$ for denoting a RHS.

* Corresponding author.
*E-mail addresses:* bellomonte@math.unipa.it (G. Bellomonte), salvatore.dibella@math.unipa.it (S. Di Bella), camillo.trapani@unipa.it (C. Trapani).

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Let $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. In $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ an involution $X \mapsto X^\dagger$ can be introduced by the equality

$$\langle X\xi|\eta\rangle = \overline{\langle X^\dagger\eta|\xi\rangle}, \quad \forall \xi,\eta \in \mathcal{D}.$$

Hence $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ is a *-invariant vector space. As we shall see, $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ can be made into a partial *-algebra by selecting an appropriate family of intermediate spaces (*interspaces*) between $\mathcal{D}$ and $\mathcal{D}^\times$ and, for this reason, this paper is a continuation of the study on the spectral properties of locally convex quasi *-algebras or partial *-algebras on which several results of a certain interest have been recently obtained, see e.g. [3,7,6,9,10,17,32,33].

The problem we want to face in this paper is that of giving a reasonable notion of spectrum of an operator $X \in \mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$; where *reasonable* means that it gives sufficient information on the behavior of the operator. Indeed, we propose a definition of resolvent set which is closely linked to the intermediate structure of interspaces that can be found between $\mathcal{D}$ and $\mathcal{D}^\times$, as it happens in many concrete examples: a spectral analysis can be performed each time we fix one of these families. Actually, the definition of resolvent set we will give depends on the choice of a family $\mathfrak{F}_0$ of interspaces. This is not a major problem if we take into account the problem that originated the spectral analysis of operators in Hilbert spaces. If, for instance, $A \in \mathcal{B}(\mathcal{H})$ (the C*-algebra of bounded operators in Hilbert space $\mathcal{H}$), looking for the resolvent set of $A$ simply means looking for the $\lambda$'s in $\mathbb{C}$ for which the equation

$$A\xi - \lambda\xi = \eta$$

has a unique solution $\xi \in \mathcal{H}$, for every choice of $\eta \in \mathcal{H}$, with $\xi$ depending continuously on $\eta$.

The same problem can be posed in the framework of rigged Hilbert spaces. For instance, between $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing $C^\infty$ functions, and $\mathcal{S}^\times(\mathbb{R})$, the space of tempered distributions, live many classical families of spaces like Sobolev spaces, Besov spaces, Bessel potential spaces, etc. Let us call $\mathfrak{F}_0$ one of these families. Then, finding solutions of the equation

$$X\xi - \lambda\xi = \eta,$$

with $X \in \mathfrak{L}(\mathcal{S}(\mathbb{R}),\mathcal{S}^\times(\mathbb{R}))$ should be intended in a more general sense: there exists a continuous extension of $X$ to a space $\mathcal{E} \in \mathfrak{F}_0$ where solutions of our equation do exist. The fact that $\xi \in \mathcal{E}$ means that the solution satisfies regularity conditions milder than those needed for $\xi$ to belong to $\mathcal{S}(\mathbb{R})$.

Rigged Hilbert spaces are a relevant example of *partial inner product* (Pip-) spaces [5]. A Pip-space $V$ is characterized by the fact that the inner product is defined only for *compatible pairs* of elements of the space $V$. It contains a complete lattice of subspaces (the so-called *assaying subspaces*) fully determined by the compatibility relation. An assaying subspace is nothing but an interspace, in the terminology adopted here. The point of view here is however different: we start from a RHS and look for convenient families of interspaces for which certain properties are satisfied. Nevertheless, we believe that an analysis similar to that undertaken here could also be performed in the more general framework of Pip-spaces, but this problem will not be considered here.

The paper is organized as follows. In Section 2 we collect some basic facts on rigged Hilbert spaces and operators on them. In Section 3 we introduce the resolvent and spectrum of an operator $X \in \mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$. This definition, as announced before, depends on the choice of a family $\mathfrak{F}_0$ of interspaces living between $\mathcal{D}$ and $\mathcal{D}^\times$ and the crucial assumption is that the operator $X$ extends continuously to some of them. Section 4 is devoted to elements of $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ that can be considered also as closable operators in Hilbert space. In particular, we give an extension of Gelfand theorem on the existence of generalized eigenvectors of a symmetric operator $X \in \mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ having a self-adjoint extension in the Hilbert space $\mathcal{H}$. We will prove, under the assumptions that $\mathcal{D} = \mathcal{D}^\infty(A)$, where $A$ is a self-adjoint operator in $\mathcal{H}$ having a Hilbert–Schmidt inverse, that the operator $X$ has a complete set of generalized eigenvectors without requiring, as done in Gelfand theorem, that $X$ leaves $\mathcal{D}$ invariant. Moreover, it is shown that these generalized eigenvectors all belong to a certain element of the chain of Hilbert spaces generated by $A$. Finally, in Section 5 we collect some examples.

## 2. Notations and preliminaries

For general aspects of the theory of partial *-algebras and of their representations, we refer to the monograph [4]. For reader's convenience, however, we repeat here the essential definitions.

A partial *-algebra $\mathfrak{A}$ is a complex vector space with conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A}\times\mathfrak{A}$, satisfying the property that $(x,y)\in\Gamma$ if, and only if, $(y^*,x^*)\in\Gamma$ and $(x\cdot y)^* = y^*\cdot x^*$. From now on, we will write simply $xy$ instead of $x\cdot y$ whenever $(x,y)\in\Gamma$. For every $y\in\mathfrak{A}$, the set of left (resp. right) multipliers of $y$ is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x\in\mathfrak{A}\colon (x,y)\in\Gamma\}$ (resp. $R(y) = \{x\in\mathfrak{A}\colon (y,x)\in\Gamma\}$). We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$. In general, a partial *-algebra is not associative.

The *unit* of partial *-algebra $\mathfrak{A}$, if any, is an element $e\in\mathfrak{A}$ such that $e = e^*$, $e\in R\mathfrak{A}\cap L\mathfrak{A}$ and $xe = ex = x$, for every $x\in\mathfrak{A}$.

Let $\mathcal{D}[t] \subset \mathcal{H} \subset \mathcal{D}^\times[t^\times]$ be a RHS and $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$.

To every $X \in \mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$ there corresponds a separately continuous sesquilinear form $\theta_X$ on $\mathcal{D}\times\mathcal{D}$ defined by

$$\theta_X(\xi,\eta) = \langle X\xi|\eta\rangle, \quad \xi,\eta\in\mathcal{D}. \tag{2}$$