J. Math. Anal. Appl. 411 (2014) 953–959

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Infinitely many small solutions for a modified nonlinear Schrödinger equations ☆

Fen Zhou [a], Ke Wu [b,c,*]

[a] Department of Information Science and Technology, Zhaotong University, Zhaotong, Yunnan 657000, PR China
[b] Department of Mathematics, Zhaotong University, Zhaotong, Yunnan 657000, PR China
[c] Department of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, PR China

ARTICLE INFO

Article history:
Received 27 March 2013
Available online 3 October 2013
Submitted by Steven G. Krantz

Keywords:
Quasi-linear
Small solutions
Dual approach
Schrödinger equations

ABSTRACT

In the present paper, we study the Modified Nonlinear Schrödinger Equations (MNSE). Without any growth condition on the nonlinear term, we obtain the existence of infinitely many small solutions for MNSE by a dual approach.

1. Introduction and main results

In this paper, we study the following Modified Nonlinear Schrödinger Equations (MNSE)

$$\begin{cases} -\Delta u - u\Delta(u^2) = g(x,u), & \text{a.e. in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and Ω is a bounded smooth domain in $\mathbb{R}^N$.

Eq. (1.1) appears in several physical models such as the superfluid film equation in plasma physics (for example, see [6–8, 11,14]). For further physical motivations and more detailed information dealing with applications, we refer readers to [3] and its references. Corresponding study in mathematics was first carried out in [12]. Due to the work in [12], it has received much attention in mathematical analysis and applications in the past ten years (for example, see [2,9,13,15]).

In general, the problem (1.1) has a variational functional defined on $H_0^1(\Omega)$ of the form

$$J(u) = \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx,$$

where $G(x,u) = \int_0^u g(x,s)\,ds$ is the primitive of function $g(x,\cdot)$.

Obviously, if $J \in C^1(H_0^1(\Omega),\mathbb{R})$, then a critical point $u \in H_0^1(\Omega)$ of J with $\int_\Omega u^2|\nabla u|^2\,dx < \infty$ is a weak solution of (1.1). However, the functional J is not always well defined for all $u \in H_0^1(\Omega)$, which means $J \in C^1(H_0^1(\Omega),\mathbb{R})$ does not always

☆ This work was supported partially by the National Natural Science Foundation of China (11261070), the Youth Program of Yunnan Provincial Science and Technology Department (2012FD062) and the Foundation of Education Department of Yunnan Province (2012Z123C).

* Corresponding author.
E-mail address: wuke2002@126.com (K. Wu).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.09.058

hold. To overcome this difficulty, several techniques were developed, including the constrained minimization argument [13], the method of a change of variables [2] (dual approaches), the perturbation method [9] and the Nehari method [15].

We devote this paper to establishing two results of existence of the infinitely many small solutions for Eq. (1.1) by the techniques introduced by Liu, Wang and Wang in [10]. To the best of our knowledge, there is no such result in current literature. We also would like to point out that, although a suitable growth condition on the nonlinear term $g(x,u)$ was regarded to be necessary for searching for solutions of (1.1), we do not need this restriction to obtain our results.

We first list the following assumptions before stating our main results.

(g_1) There exists a constant $\delta > 0$ such that $g(x,-t) = -g(x,t)$ for $|t| \leqslant \delta$ and all $x \in \Omega$.
(g_2) $\lim_{t\to 0} \frac{g(x,t)}{t} = +\infty$ uniformly for $x \in \Omega$.
(g_3) There exist constants $r > 0$ and $\alpha \in (0,2)$ such that $g(x,t)t \leqslant \alpha G(x,t)$ for $|t| \leqslant r$ and all $x \in \Omega$.

Theorem 1.1. *Suppose that (g_1)–(g_3) are satisfied. Then (1.1) admits a weak solution sequence $\{u_n\}$ such that $u_n \neq 0$, $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $J(u_n) < 0$, $J(u_n) \to 0^-$ and $u_n \to 0$ in $H_0^1(\Omega)$.*

Remark 1.1. In order to obtain a small solution sequence of (1.1), the condition (g_3) is not essential. In fact, without (g_3), we also have the following Theorem 1.2. By a simple comparison between Theorem 1.1 and Theorem 1.2, we find that the energy of weak solutions obtained in Theorem 1.1 is negative, but we do not know whether the same fact is true in Theorem 1.2. Consequently, the condition (g_3) ensures the existence of negative energy small solution of (1.1).

Theorem 1.2. *Suppose that (g_1) and (g_2) are satisfied. Then either* (i) *or* (ii) *below holds.*

(i) The conclusion of Theorem 1.1 holds.
(ii) (1.1) has a weak solution sequence $\{u_n\}$ such that $u_n \neq 0$, $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $J(u_n) = 0$, and $u_n \to 0$ in $H_0^1(\Omega)$.

2. Preliminaries

In this section, we introduce a variational structure of problem (1.1). We employ an argument developed in [10]. Precisely, we make a change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2|f(t)|^2}} \quad \text{on } t \in [0,+\infty)$$

and

$$f(-t) = -f(t) \quad \text{on } t \in (-\infty, 0].$$

Lemma 2.1. *The function $f(t)$ enjoys the following properties*:

(f_1) *f is uniquely defined C^∞ function and invertible*;
(f_2) $0 < f'(t) \leqslant 1$, $\forall t \in \mathbb{R}$;
(f_3) $|f(t)| \leqslant |t|$, $\forall t \in \mathbb{R}$;
(f_4) $\lim_{t\to 0} \frac{f(t)}{t} = 1$;
(f_5) $\frac{f(t)}{2} \leqslant tf'(t) \leqslant f(t)$, $\forall t \geqslant 0$ *and* $f(t) \leqslant tf'(t) \leqslant \frac{f(t)}{2}$, $\forall t \leqslant 0$;
(f_6) *There exists a positive constant $C < 1$ such that*

$$|f(t)| \geqslant \begin{cases} C|t|, & |t| \leqslant 1, \\ C|t|^{\frac{1}{2}}, & |t| \geqslant 1. \end{cases}$$

The properties of the function f have been listed by many authors (for example, see Lemma 2.1 in [4]). We omit their proofs here.

Consider the functional defined on $E = H_0^1(\Omega)$ by

$$I(v) := J\big(f(v)\big) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G\big(x, f(v)\big)\,dx$$

for all $v \in E$. Then $I(\cdot)$ is well defined on E and $I \in C^1(E,\mathbb{R})$ if and only if $\int_\Omega G(x, f(\cdot))\,dx$ has the same property as functional I. Hence, if $\int_\Omega G(x, f(\cdot))\,dx$ is continuously differential on E, then $I \in C^1(E,\mathbb{R})$,