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Homoclinic orbits and periodic solutions for a class of Hamiltonian systems on time scales

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ABSTRACT

In this paper, we are concerned with a second order non-autonomous Hamiltonian system on time scales $\mathbb{T}$

$$u^{\Delta\Delta}\big(\rho(t)\big) + V_u\big(t, u(t)\big) = f(t), \quad t \in \mathbb{T}^{\kappa}.$$

Under certain conditions, the existence and multiplicity of periodic solutions are obtained for this Hamiltonian system on time scales by using the saddle point theory, the least action principle as well as the three-critical-point theorem. In addition, the existence of homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a given sequence of Hamiltonian system on time scales by means of the mountain pass theorem and the standard minimizing argument.

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers, which has the topology inherited from the real numbers with the standard topology. Hilger [12] presented the theory of time scales with the motivation of providing a unified approach to discrete and continuous analysis. In the past decades, there has been an increasing interest in the study of dynamic equations on time scales [1,4,11,17,19,23,26–34], and time scales have become a crucial role in various equations and systems arising in astronomy and biology, being particularly relevant in ecology, where homoclinic orbits of dynamical systems on a time scale have been recognized to contribute critically to the stable or unstable outcome of models of biological populations [4,5,9,23,25]. For example, they can model insect populations that evolve continuously while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population [4]. Therefore, theoretically and practically it is potentially helpful to study equations and systems on time scales for their richer and more plausible dynamics.

Although considerable attention has been dedicated to homoclinic orbits for continuous or discrete Hamiltonian systems, see [7,8,14,15,21,24,35–37] and the references therein. To the best of our knowledge, there is few work on homoclinic orbits for Hamiltonian systems on time scales. One of interesting and open problems on dynamic equations on time scales is to investigate positive solutions of discrete or continuous Hamiltonian systems on time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). Nonempty closed subsets of the reals are considered to be time scales and quite a few sufficient conditions of the existence and multiplicity of solutions for Hamiltonian systems have been presented [2,13,30,32,38].

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To address the motivation and goal of the study on homoclinic orbits and periodic solutions of Hamiltonian systems on time scales, we start with a brief review on some related results in the existing literature. In [21], Rabinowitz considered a second order continuous Hamiltonian system

$$\ddot{u}+V_u(t,u)=0, \tag{1}$$

where $u\in\mathbb{R}^n$ and $V\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ is T-periodic in t. Under certain assumptions on V, the homoclinic orbit u is obtained as the limit of $2kT$-periodic solutions u_k of system (1) as $k\to\infty$. The subharmonic u_k is obtained in turn via the mountain pass theorem. In [14,15], Izydorek and Janczewska dealt with the existence of homoclinic orbits of system (1) with the forcing term. The novelties are a relaxed superquadratic assumption on the time-periodic potential $V(t,u)$, and the addition of a small but possibly aperiodic forcing term $f(t)$ to the equation.

In [32], Su and Li considered a non-autonomous second order Hamiltonian system

$$\begin{cases} u^{\Delta^2}\big(\rho(t)\big)=\nabla H\big(t,u(t)\big), & \Delta\text{-a.e. } t\in[0,T]_{\mathbb{T}},\\ u(0)-u(T)=u^{\Delta}\big(\rho(0)\big)-u^{\Delta}\big(\rho(T)\big)=0, \end{cases} \tag{2}$$

and established the existence of periodic solutions of system (2) on time scales $\mathbb{T}$. A distinctive feature lies in that the integral used therein is not Hilger's integral but a new integral on time scales $\mathbb{T}$. On the base of Hilger's integral, Zhou and Li [38] explored Sobolev's spaces on time scales. As an application, they investigated the existence of solutions for the following second order Hamiltonian systems on time scales $\mathbb{T}$

$$\begin{cases} u^{\Delta^2}(t)=\nabla F\big(\sigma(t),u^{\sigma}(t)\big), & \Delta\text{-a.e. } t\in[0,T]_{\mathbb{T}^{\kappa}},\\ u(0)-u(T)=0, \quad u^{\Delta}(0)-u^{\Delta}(T)=0. \end{cases}$$

Su and Feng [30] applied the variational method and the critical point theory to study the same problem and established the existence and multiplicity of periodic solutions for this Hamiltonian system.

In this paper, we consider a second order non-autonomous Hamiltonian system on time scales $\mathbb{T}$:

$$u^{\Delta\Delta}\big(\rho(t)\big)+V_u(t,u)=f(t), \tag{3}$$

where $t\in\mathbb{T}^{\kappa}$, $u\in\mathbb{R}^n$, $V:\mathbb{T}\times\mathbb{R}^n\to\mathbb{R}$ and $f:\mathbb{T}\to\mathbb{R}^n$. When $V(t,x)=-K(t,x)+W(t,x)$ is measurable in t for every $x\in\mathbb{R}^n$ and continuously differentiable in x for $t\in\mathbb{T}$, both $K(t,x)$ and $W(t,x)$ are T-periodic in t, and $f(t)$ is a T-periodic bounded function, we obtain the existence of at least one periodic solution by using the saddle point theorem and the least action principle, and obtain at least three distinct periodic solutions via the three-critical-point theorem. These results are sharp even for the associated differential ($\mathbb{T}=\mathbb{R}$) and difference equations ($\mathbb{T}=\mathbb{Z}$). When V is C^1_{Δ}-smooth and T-periodic functions with respect to t, a homoclinic orbit of Hamiltonian system (3) is obtained as a limit of $2kT$-periodic solutions of a certain sequence of Hamiltonian system on time scales by developing the mountain pass theorem and the standard minimizing argument.

We say that a solution u of Hamiltonian system (3) is homoclinic to zero if it satisfies $u(t)\to 0$ as $t\to\pm\infty$, where $t\in\mathbb{T}$. In addition, if $u\neq 0$, then u is called a nontrivial homoclinic solution.

The paper is outlined as follows. In Section 2, we introduce some background information on the delta derivative of f from $\mathbb{T}\to\mathbb{R}^N$ and the related definitions. In Section 3, we present some technical lemmas. In Section 4, we show the variational structure of our Hamiltonian system, and establish the existence and multiplicity of its periodic solutions. In the subsequent paper, we prove the existence of homoclinic orbit of the second order Hamiltonian system (3) on time scales.

2. Preliminaries

To make this paper self-contained and state our discussions in a straightforward way, in this section we present some basic definitions and the related propositions [4,10,12,16,22,30,32,38] which may help us better understand our main results and proofs described in Sections 4 and 5. For the terminologies such as the measure on time scales, absolute continuity on time scales, and fundamental properties of Sobolev's spaces on time scales, we refer the reader to [16,22,38] and the references therein.

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. If $\mathbb{T}$ has a right-scattered minimum m, we define $\mathbb{T}_{\kappa}=\mathbb{T}-\{m\}$; otherwise, we set $\mathbb{T}_{\kappa}=\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, we define $\mathbb{T}^{\kappa}=\mathbb{T}-\{M\}$; otherwise, we set $\mathbb{T}^{\kappa}=\mathbb{T}$. The forward graininess is $\mu(t):=\sigma(t)-t$. Similarly, the backward graininess is $\nu(t):=t-\rho(t)$.

If $f:\mathbb{T}\to\mathbb{R}$ is a function and $t\in\mathbb{T}^{\kappa}$, the delta derivative of f at the point t is defined by the number $f^{\Delta}(t)$ (provided it exists) with the property that for any $\epsilon>0$, there is a neighborhood $U\subset\mathbb{T}$ of t such that

$$\big|f\big(\sigma(t)\big)-f(s)-f^{\Delta}(t)\big(\sigma(t)-s\big)\big|\leqslant\epsilon\big|\sigma(t)-s\big|$$

for all $s\in U$.