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Asymptotic behavior of least energy solutions for a 2D nonlinear Neumann problem with large exponent

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ABSTRACT

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p)\quad \begin{cases} -\Delta u + u = 0 & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number.

We study the asymptotic behavior of least energy solutions to (E_p) when the nonlinear exponent p gets large. Following the arguments of X. Ren and J.C. Wei [13,14], we show that the least energy solutions remain bounded uniformly in p, and it develops one peak on the boundary, the location of which is controlled by the Green function associated to the linear problem.

1. Introduction

In this paper, we consider the following elliptic problem with the nonlinear Neumann boundary condition:

$$(E_p)\quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^p & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, ν is the outer unit normal vector to $\partial\Omega$, and $p > 1$ is any positive number. Let $H^1(\Omega)$ be the usual Sobolev space with the norm $\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + u^2)\,dx$. Since the trace Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$ is compact for any $p > 1$, we can obtain at least one solution of (1.1) by a standard variational method. In fact, let us consider the constrained minimization problem

$$C_p^2 = \inf\left\{ \int_\Omega \left(|\nabla u|^2 + u^2\right) dx \,\middle|\, u \in H^1(\Omega),\ \int_{\partial\Omega} |u|^{p+1}\, ds_x = 1 \right\}. \tag{1.2}$$

Standard variational method implies that C_p^2 is achieved by a positive function $\bar{u}_p \in H^1(\Omega)$ and then $u_p = C_p^{2/(p-1)}\bar{u}_p$ solves (1.1). We call u_p a least energy solution to the problem (1.1).

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In this paper, we prove the followings:

Theorem 1. *Let u_p be a least energy solution to (E_p). Then it holds*

$$1 \leqslant \liminf_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leqslant \limsup_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} \leqslant \sqrt{e}.$$

To state further results, we set

$$v_p = u_p \Big/ \left(\int_{\partial\Omega} u_p^p \, ds_x \right). \tag{1.3}$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Then for any sequence v_{p_n} of v_p defined in (1.3) with $p_n \to \infty$, there exists a subsequence (still denoted by v_{p_n}) and a point $x_0 \in \partial\Omega$ such that the following statements hold true.*

(1)

$$f_n = \frac{u_{p_n}^{p_n}}{\int_{\partial\Omega} u_{p_n}^{p_n} \, ds_x} \overset{*}{\rightharpoonup} \delta_{x_0}$$

in the sense of Radon measures on $\partial\Omega$.

(2) $v_{p_n} \to G(\cdot, x_0)$ *in $C^1_{loc}(\bar{\Omega} \setminus \{x_0\})$, $L^t(\Omega)$ and $L^t(\partial\Omega)$ respectively for any $1 \leqslant t < \infty$, where $G(x, y)$ denotes the Green function of $-\Delta$ for the following Neumann problem*:

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = 0 & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu_x}(x, y) = \delta_y(x) & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

(3) x_0 *satisfies*

$$\nabla_{\tau(x_0)} R(x_0) = \vec{0},$$

where $\tau(x_0)$ denotes a tangent vector at the point $x_0 \in \partial\Omega$ and R is the Robin function defined by $R(x) = H(x, x)$, where

$$H(x, y) := G(x, y) - \frac{1}{\pi} \log|x - y|^{-1}$$

denotes the regular part of G.

Concerning related results, X. Ren and J.C. Wei [13,14] first studied the asymptotic behavior of least energy solutions to the semilinear problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

as $p \to \infty$, where Ω is a bounded smooth domain in $\mathbb{R}^2$. They proved that the least energy solutions remain bounded and bounded away from zero in L^∞-norm uniformly in p. As for the shape of solutions, they showed that the least energy solutions must develop one "peak" in the interior of Ω, which must be a critical point of the Robin function associated with the Green function subject to the Dirichlet boundary condition. Later, Adimurthi and Grossi [1] improved their results by showing that, after some scaling, the limit profile of solutions is governed by the Liouville equation

$$-\Delta U = e^U \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^U \, dx < \infty,$$

and obtained that $\lim_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} = \sqrt{e}$ for least energy solutions u_p. Actual existence of concentrating solutions to (1.1) is recently obtained by H. Castro [4] by a variational reduction procedure, along the line of [8] and [6]. As for construction of concentrating solutions to related problems, see also [7,9], and [11].

Also in our case, we may conjecture that the limit problem of (1.1) is

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial U}{\partial \nu} = e^U & \text{on } \partial\mathbb{R}^2_+, \\ \int_{\partial\mathbb{R}^2_+} e^U \, ds < \infty, \end{cases}$$

and $\lim_{p\to\infty} \|u_p\|_{L^\infty(\partial\Omega)} = \sqrt{e}$ holds true at least for least energy solutions u_p. Verification of these conjectures remains as the future work.