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An approach of majorization in spaces with a curved geometry

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ABSTRACT

The Hardy–Littlewood–Pólya majorization theorem is extended to the framework of some spaces with a curved geometry (such as the global NPC spaces and the Wasserstein spaces). We also discuss the connection between our concept of majorization and the subject of Schur convexity. Several applications are included.

1. Introduction

In 1929, G.H. Hardy, J.E. Littlewood and G. Pólya [13,14] have proved an important characterization of convex functions in terms of a partial ordering of vectors $x=(x_1,\dots,x_n)$ in $\mathbb{R}^n$. In order to state it we need a preparation. We denote by $x^{\downarrow}$ the vector with the same entries as x but rearranged in decreasing order,

$$x_1^{\downarrow} \geqslant \cdots \geqslant x_n^{\downarrow}.$$

Then x is *weakly majorized* by y (abbreviated, $x \prec_* y$) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=,1}^{k} y_i^{\downarrow} \quad \text{for } k=1,\dots,n \tag{1}$$

and x is *majorized* by y (abbreviated, $x \prec y$) if in addition

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}. \tag{2}$$

Intuitively, $x \prec y$ means that the components in x are less spread out than the components in y. As shown in Theorem 1 below, the concept of majorization admits an order-free characterization based on the notion of doubly stochastic matrix. Recall that a matrix $A \in \mathrm{M}_n(\mathbb{R})$ is *doubly stochastic* if it has nonnegative entries and each row and each column sums to unity.

Theorem 1. *(Hardy, Littlewood and Pólya [13, Theorem 8]) Let x and y be two vectors in $\mathbb{R}^n$, whose entries belong to an interval I. Then the following statements are equivalent*:

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i) $x \prec y$;
ii) *There is a doubly stochastic matrix* $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ *such that* $x = Ay$;
iii) *The inequality* $\sum_{i=1}^n f(x_i) \leqslant \sum_{i=1}^n f(y_i)$ *holds for every continuous convex function* $f : I \to \mathbb{R}$.

The proof of this result is also available in the recent monographs [22] and [25].

Remark 1. M. Tomić [31] and H. Weyl [32] have noticed the following characterization of weak majorization: $x \prec_* y$ if and only if $\sum_{i=1}^n f(x_i) \leqslant \sum_{i=1}^n f(y_i)$ for every continuous nondecreasing convex function f defined on an interval containing the components of x and y. The reader will find the details in [22, Proposition B2, p. 157].

Nowadays many important applications of majorization to matrix theory, numerical analysis, probability, combinatorics, quantum mechanics etc. are known, see [3,22,25,27,28]. They were made possible by the constant growth of the theory, able to uncover the most diverse situations.

In what follows we will be interested in a simple but basic extension of the concept of majorization as mentioned above: the weighted majorization. Indeed, the entire subject of majorization can be switched from vectors to Borel probability measures by identifying a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ with the discrete measure $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ acting on $\mathbb{R}$. By definition,

$$\frac{1}{n}\sum_{i=1}^n \delta_{x_i} \prec \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$$

if the conditions (1) and (2) above are fulfilled, and Theorem 1 can be equally seen as a characterization of this instance of majorization.

Choquet's theory made available a very general framework of majorization by allowing the comparison of Borel probability measures whose supports are contained in a compact convex subset of a locally convex separated space. The highlights of this theory are presented in [28] and refer to a concept of majorization based on condition iii) in Theorem 1 above. The particular case of discrete probability measures on the Euclidean space $\mathbb{R}^N$, that admits an alternative approach via condition ii) in the same Theorem 1 is of interest to us. Indeed, in this case one can introduce a relation of the form

$$\sum_{i=1}^m \lambda_i \delta_{x_i} \prec \sum_{j=1}^n \mu_j \delta_{y_j}, \tag{3}$$

where all coefficients λ_i and μ_j are weights, by asking the existence of an $m \times n$-dimensional matrix $A = (a_{ij})_{i,j}$ such that

$$a_{ij} \geqslant 0, \quad \text{for all } i, j, \tag{4}$$

$$\sum_{j=1}^n a_{ij} = 1, \quad i = 1, \dots, m, \tag{5}$$

$$\mu_j = \sum_{i=1}^m a_{ij}\lambda_i, \quad j = 1, \dots, n \tag{6}$$

and

$$x_i = \sum_{j=1}^n a_{ij} y_j, \quad i = 1, \dots, n. \tag{7}$$

The matrices verifying the conditions (4) and (5) are called *stochastic on rows*. When $m = n$ and all weights λ_i and μ_j are equal, the condition (6) assures the *stochasticity on columns*, so in that case we deal with doubly stochastic matrices.

The fact that (3) implies

$$\sum_{i=1}^m \lambda_i f(x_i) \prec \sum_{j=1}^n \mu_j f(y_j),$$

for every continuous convex function f defined on a convex set containing all points x_i and y_i, is covered by a general result due to S. Sherman [29]. See also the paper of J. Borcea [7] for a nice proof and important applications.

It is worth noticing that the extended definition of majorization given by (3) is related, via equality (7), to an optimization problem as follows:

$$x_i = \arg\min_{z \in \mathbb{R}^N} \frac{1}{2}\sum_{j=1}^n a_{ij}\|z - y_j\|^2, \quad \text{for } i = 1, \dots, m.$$