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Estimates for multiparameter maximal operators of Schrödinger type

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ABSTRACT

Multiparameter maximal estimates are considered for operators of Schrödinger type. Sharp and almost sharp results, that extend work by Rogers and Villarroya, are obtained. We provide new estimates via the integrability of the kernel which naturally appears with a TT^* argument and discuss the behavior at the endpoints. We treat in particular the case of global integrability of the maximal operator on finite time for solutions to the linear Schrödinger equation and make some comments on an open problem.

1. Introduction and main results

Assuming $a > 1$ and letting f belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, we set

$$S_t f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \widehat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Here $\widehat{f}$ denotes the Fourier transform of the function f, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We also set $u(x,t) = (2\pi)^{-n} S_t f(x)$. It then follows that $u(x,0) = f(x)$ and in the case $a = 2$, the function u satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$. Also, more generally, if $a = 2k$ for some $k = 1, 2, 3, \dots$, then u satisfies the equation $i\partial u/\partial t = \Delta^k u$, if k is odd and $i\partial u/\partial t = -\Delta^k u$, if k is even.

We shall study the maximal function $S^* f$ defined by

$$S^* f(x) = \sup_{0<t<1} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n,$$

and define Sobolev spaces H_s by setting

$$H_s = \left\{ f \in \mathcal{S}' \colon \|f\|_{H_s} < \infty \right\}, \quad s \in \mathbb{R},$$

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where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

The homogeneous Sobolev spaces $\dot{H}_s$, for $s \in \mathbb{R}$, are defined by

$$\dot{H}_s = \left\{ f \in \mathcal{S}' \colon \|f\|_{\dot{H}_s} < \infty \right\},$$

where

$$\|f\|_{\dot{H}_s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

The inequality

$$\left\|S^* f\right\|_{L^2(B)} \leqslant C\|f\|_{H_s}, \tag{1}$$

for arbitrary balls B has been studied by several authors. In the case $n = 1$, it is known that (1) holds if and only if $s \geqslant 1/4$ (see Carleson [2], Dahlberg and Kenig [3], and Sjölin [10]). In the case $n = 2$ and $a = 2$, Lee [8], extending previous results in [18] and [17], has proved that (1) holds for $s > 3/8$. In the case $n \geqslant 3$, Sjölin [10] and Vega [19] proved that (1) holds for $s > 1/2$.

As is well known, the inequality (1) implies that

$$\lim_{t \to 0} \frac{1}{(2\pi)^n} S_t f(x) = f(x), \quad \text{a.e.},$$

for every $f \in H_s$. The above estimates therefore give pointwise convergence results. In the case $a = 2$, Bourgain [1] has recently improved these results and proved that one has convergence almost everywhere for every $f \in H_s(\mathbb{R}^n)$ if $s > 1/2 - 1/4n$. On the other hand Bourgain has also proved that one does not have convergence almost everywhere for all $f \in H_s(\mathbb{R}^n)$ if $n \geqslant 5$ and $s < 1/2 - 1/n$.

For $n = 1$ and $a > 1$ we set $M^* f = S^* f$ and

$$M^{**} f(x) = \sup_{t \in \mathbb{R}} \left|S_t f(x)\right|, \quad x \in \mathbb{R}.$$

In harmonic analysis considerable attention has been given to multiparameter singular integrals and related operators. Some examples of this can be seen in the work of E.M. Stein and R. Fefferman [4,6,5,7]. In this paper we introduce in the same spirit multiparameter operators of Schrödinger type.

For $n \geqslant 2$ and a multiindex $a = (a_1, a_2, \ldots, a_n)$, with $a_j > 1$, and $f \in \mathcal{S}(\mathbb{R}^n)$, we now set

$$S_t f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1|\xi_1|^{a_1} + t_2|\xi_2|^{a_2} + \cdots + t_n|\xi_n|^{a_n})} \widehat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

where $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. In the remaining part of this paper, S_t will be defined in this way if $n \geqslant 2$. Finally, we will define maximal operators for $n \geqslant 2$ by letting

$$M^* f(x) = \sup_{0 < t_i < 1} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n,$$

and

$$M^{**} f(x) = \sup_{t_i \in \mathbb{R}} \left|S_t f(x)\right|, \quad x \in \mathbb{R}^n.$$

In this paper we will study the inequality

$$\left\|M^{**} f\right\|_q \leqslant C\|f\|_{\dot{H}_s}, \tag{2}$$

as well as

$$\left\|M^* f\right\|_q \leqslant C\|f\|_{H_s}, \tag{3}$$

for different values of $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$ and the multiindex a. Here we shall use the notation $\|\cdot\|_q = \|\cdot\|_{L^q(\mathbb{R}^n)}$.

We can state the following results. The first two theorems are concerned with the case $n = 1$. Some parts of them are already known, but we bring them here for the sake of completeness.