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The numerical range and the spectrum of a product of two orthogonal projections

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ABSTRACT

The aim of this paper is to describe the closure of the numerical range of the product of two orthogonal projections in Hilbert space as a closed convex hull of some explicit ellipses parametrized by points in the spectrum. Several improvements (removing the closure of the numerical range of the operator, using a parametrization after its eigenvalues) are possible under additional assumptions. An estimate of the least angular opening of a sector with vertex 1 containing the numerical range of a product of two orthogonal projections onto two subspaces is given in terms of the cosine of the Friedrichs angle. Applications to the rate of convergence in the method of alternating projections and to the uncertainty principle in harmonic analysis are also discussed.

1. Introduction

Background. The numerical range of a Hilbert space operator $T \in \mathcal{B}(H)$ is defined as $W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}$. It is always a convex set in the complex plane (the Toeplitz–Hausdorff theorem) containing in its closure the spectrum of the operator. Also, the intersection of the closure of the numerical ranges of all the operators similar to T is precisely the convex hull of the spectrum of T (Hildebrandt's theorem). We refer to the book [16] for these and other facts about numerical ranges. Another useful property the numerical ranges have is the following recent result of Crouzeix [8]: for every $T \in \mathcal{B}(H)$ and every polynomial p, we have $\|p(T)\| \leqslant 12 \sup_{z\in W(T)} |p(z)|$.

The problem. The main aim of this paper is to study the numerical range $W(T)$ and the numerical radius, defined by $\omega(T) = \sup\{|z|,\ z \in W(T)\}$, of a product of two orthogonal projections $T = P_{M_2}P_{M_1}$. In what follows we denote by P_M the orthogonal projection onto the closed subspace M of a given Hilbert space H. We prove a representation of the closure of $W(T)$ as a closed convex hull of some explicit ellipses parametrized by points in the spectrum $\sigma(T)$ of T and we discuss several applications. We also study the relationship between the numerical range (numerical radius) of a product of two orthogonal projections and its spectrum (resp. spectral radius). Recall that the spectral radius $r(T)$ of $T \in \mathcal{B}(H)$ is defined as $r(T) = \sup\{|z|,\ z \in \sigma(T)\}$.

Previous results. Orthogonal projections in Hilbert space are basic objects of study in Operator theory. Products or sums of orthogonal projections, in finite or infinite dimensional Hilbert spaces, appear in various problems and in many different areas, pure or applied. We refer the reader to a book [14] and two recent surveys [5,15] for more information. The fact that the numerical range of a finite product of orthogonal projections is included in some sector of the complex plane with vertex at 1 was an essential ingredient in the proof by Delyon and Delyon [10] of a conjecture of Burkholder, saying that the iterates of a product of conditional expectations are almost surely convergent to some conditional expectation in an L^2

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space (see also [7,9]). For a product of two orthogonal projections we know that the numerical range is included in a sector with vertex one and angle $\pi/6$ [9].

The spectrum of a product of two orthogonal projections appears naturally in the study of the rate of convergence in the strong operator topology of $(P_{M_2}P_{M_1})^n$ to $P_{M_1\cap M_2}$ (cf. [1–4,11–13]). This is a particular instance of von Neumann–Halperin type theorems, sometimes called in the literature the method of alternating projections. The following dichotomy holds (see [1]): either the sequence $(P_{M_2}P_{M_1})^n$ converge uniformly with an exponential speed to $P_{M_1\cap M_2}$ (if $1\notin\sigma(P_{M_2}P_{M_1})$), or the sequence of alternating projections $(P_{M_2}P_{M_1})^n$ converges arbitrarily slowly in the strong operator topology (if $1\in\sigma(P_{M_2}P_{M_1})$). We refer to [2,3] for several possible meanings of "slow convergence".

An occurrence of the numerical range of operators related to sums of orthogonal projections appears also in some Harmonic analysis problems. The uncertainty principle in Fourier analysis is the informal assertion that a function $f\in L_2(\mathbb{R})$ and its Fourier transform $\mathcal{F}(f)$ cannot be too small simultaneously. Annihilating pairs and strong annihilating pairs are a way to formulate this idea (precise definitions will be given in Section 5). Characterizations of annihilating pairs and strong annihilating pairs (S,Σ) in terms of the numerical range of the operator $P_S+\mathrm{i}P_\Sigma$, constructed using some associated orthogonal projections P_S and P_Σ, can be found in [18,21].

Main results. Our first contribution is an exact formula for the closure of the numerical range $\overline{W(P_{M_2}P_{M_1})}$, expressed as a convex hull of some ellipses $\mathscr{E}(\lambda)$, parametrized by points in the spectrum $(\lambda\in\sigma(P_{M_2}P_{M_1}))$.

Definition 1.1. Let $\lambda\in[0,1]$. We denote $\mathscr{E}(\lambda)$ the domain delimited by the ellipse with foci 0 and λ, and minor axis length $\sqrt{\lambda(1-\lambda)}$.

We refer to Remark 3.3 and to Fig. 1 for more information about these ellipses.

Theorem 1.2. *Let M_1 and M_2 be two closed subspaces of H such that $M_1\neq H$ or $M_2\neq H$. Then the closure of the numerical range of $P_{M_2}P_{M_1}$ is the closure of the convex hull of the ellipses $\mathscr{E}(\lambda)$ for $\lambda\in\sigma(P_{M_2}P_{M_1})$, i.e.:*

$$\overline{W(P_{M_2}P_{M_1})}=\overline{\operatorname{conv}\bigl\{\cup_{\lambda\in\sigma(P_{M_2}P_{M_1})}\mathscr{E}(\lambda)\bigr\}}.$$

The proof uses in an essential way Halmos' two subspaces theorem recalled in the next section. We will use a completely different approach to describe the numerical range (without the closure) of $T=P_{M_2}P_{M_1}$ under the additional assumption that the self adjoint operator $T^*T=P_{M_1}P_{M_2}P_{M_1}$ is diagonalizable (see Definition 3.7). In this case the numerical range $W(T)$ is the convex hull of the same ellipses as before but this time parametrized by the point spectrum $\sigma_p(T)$ (= eigenvalues) of $T=P_{M_2}P_{M_1}$.

Theorem 1.3. *Let H be a separable Hilbert space. Let M_1 and M_2 be two closed subspaces of a Hilbert space H such that $M_1\neq H$ or $M_2\neq H$. If $P_{M_1}P_{M_2}P_{M_1}$ is diagonalizable, then the numerical range $W(P_{M_2}P_{M_1})$ is the convex hull of the ellipses $\mathscr{E}(\lambda)$, with the λ's being the eigenvalues of $P_{M_2}P_{M_1}$, i.e.:*

$$W(P_{M_2}P_{M_1})=\operatorname{conv}\bigl\{\cup_{\lambda\in\sigma_p(P_{M_2}P_{M_1})}\mathscr{E}(\lambda)\bigr\}.$$

Concerning the relationship between the numerical radius and the spectral radius of a product of two orthogonal projections we prove the following result.

Proposition 1.4. *Let M_1, M_2 be two closed subspaces of H. The numerical radius and the spectral radius of $P_{M_2}P_{M_1}$ are linked by the following formula*:

$$\omega(P_{M_2}P_{M_1})=\frac{1}{2}\Bigl(\sqrt{r(P_{M_2}P_{M_1})}+r(P_{M_2}P_{M_1})\Bigr).$$

The proof is an application of Theorem 1.2 and the obtained formula is better than Kittaneh's inequality [19] whenever the Friedrichs angle (Definition 2.7) between M_1 and M_2 is positive.

Theorems 1.2 and 1.3 can be used to localize $W(P_{M_2}P_{M_1})$ even if the spectrum of $P_{M_2}P_{M_1}$ is unknown. We mention here the following important consequence about the inclusion of $W(P_{M_2}P_{M_1})$ in a sector of vertex 1 whose angular opening is expressed in terms of the cosine of the Friedrichs angle $\cos(M_1,M_2)$ between the subspaces M_1 and M_2. This is a refinement of the Crouzeix's result [9] for products of two orthogonal projections.

Proposition 1.5. *Let M_1 and M_2 be two closed subspaces of a Hilbert space H. We have the following inclusion*:

$$W(P_{M_2}P_{M_1})\subset\left\{z\in\mathbb{C},\ \bigl|\arg(1-z)\bigr|\leqslant\arctan\left(\sqrt{\frac{\cos^2(M_1,M_2)}{4-\cos^2(M_1,M_2)}}\right)\right\}.$$