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A new set of solutions to a singular second-order differential equation arising in boundary layer theory

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ABSTRACT

The paper studies a boundary value problem of a nonlinear second-order differential equation that governs the laminar flow of Newtonian and non-Newtonian fluids. It presents a new sufficient condition for the existence of solutions to the boundary value problem.

1. Introduction

Assume that a flat plate with a porous surface is moving at a constant speed U_w in the direction parallel to a uniform flow of another constant speed U_∞. If the same incompressible fluid is injected or sucked with a velocity V_w through the surface of the plate, then the laminar flow is generated near the plate and forms a boundary layer. Assume also that the body force, external pressure gradients and the viscous dissipation do not exist or are negligible. Then, the laminar flow is governed by a nonlinear third order differential equation

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \tag{1}$$

with boundary conditions

$$f(0) = -C, \qquad f'(0) = \xi, \qquad \lim_{\eta\to\infty} f'(\eta) = 1, \tag{2}$$

where η is the similarity variable, $f(\eta)$ is the dimensionless stream function, $\xi = U_w/U_\infty$, and C is a parameter related to suction if it is negative or injection if positive. The case $C = 0$ means that there is neither injection nor suction. If $\xi = 0$, then the plate does not move. When $\xi > 0$, the plate and the stream move in the same direction. When $\xi < 0$, they move in opposite directions. Eq. (1) with (2) for $C = \xi = 0$ is the classical Blasius equation. The first analytical proof of existence and uniqueness of the solution to the Blasius equation was established by Weyl [19] in 1942.

Using the Crocco transformation [3], one can write the boundary value problem (1), (2) as a second-order nonlinear equation

$$g(x)g''(x) + x = 0, \quad \xi < x < 1, \tag{3}$$

with boundary conditions

$$g'(\xi) = C, \qquad g(1) = 0. \tag{4}$$

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Before 2010, all of the theoretical results on the existence of solutions to Eqs. (3) and (4) required ξ to be bounded below [2,8,9,14]. In 2010, Lu [12] proved a new result which removes the lower boundedness requirement on ξ for the existence of solutions of Eq. (3) with (4). This paper continues the work of [12].

A general form of Eq. (3), introduced by Soewono et al. [17,18] in 1991, is

$$g(x)g''(x) + h(x) = 0, \quad \xi < x < 1, \tag{5}$$

where $h(x)$ is a continuous and increasing function satisfying

$$M_1|x| \leqslant \left|h(x)\right| \leqslant M_2|x|, \tag{6}$$

for $\xi \leqslant x \leqslant 1$, and

$$\lim_{x\to\infty} \frac{h(x)}{x} > M \tag{7}$$

for some positive constants M_1, M_2 and M. Soewono et al. proved the existence of solutions to (5), (4) and required ξ to be bounded below.

If the fluid is non-Newtonian (for example, pseudo-plastic) and obeys the power law, the similarity equation can be written as

$$\left\{\left[f''(\eta)\right]^N\right\}' + f(\eta)f''(\eta) = 0, \tag{8}$$

where $0 < N < 1$. Setting $x = f'(\eta)$ and $g(x) = [f''(\eta)]^N$ in Eq. (8) gives [1,16]

$$g^{1/N}(x)g''(x) + x = 0, \quad \xi < x < 1. \tag{9}$$

Eqs. (8) and (9) have been studied by Zheng and He, and Lu [10–13,20]. Their results still require that ξ must be bounded below for the existence of solutions of Eq. (8) with (2).

This paper studies a general form of (9),

$$g^{1/N}(x)g''(x) + h(x) = 0, \quad \xi < x < 1, \tag{10}$$

where $h(x)$ is a continuous and increasing function satisfying (6) for $x \in R$. It presents a new sufficient condition for the existence of multiple solutions of Eq. (10) with (4) for $\xi, C < 0$ and for $0 < N \leqslant 1$. It also proves the non-existence of the solution to the boundary value problem for some $C > 0$ and $\xi < 0$. Recent results of numerical solutions to the boundary value problem can be found from other publications [5,7]. The main results of this paper are the following two theorems.

Theorem 1. *For any $\xi < 0$ there exists an interval $(C_1, C_2) \subset (-\infty, 0)$ depending on ξ, such that for $C \in (C_1, C_2)$ the boundary value problem (10), (4) admits at least two nonnegative solutions.*

Theorem 2. *If $\xi < 0$, $C > 0$, and $|C\xi| > (2^{1/N}M_2/3)^{N/(1+N)}$, where M_2 is a constant given in (6), then the boundary value problem (10), (4) does not have a solution.*

Remark 1. The asymptotic condition (7) is not needed in this paper. From Theorem 2, we see that if C is fixed, then ξ must be bounded, which agrees with the known results. This paper only focuses on the existence of multiple solutions of Eq. (10) with (4). The uniqueness of the solution of (10), (4) for $\xi \geqslant 0$ is omitted in this paper, since its proof is similar to that in [13,20], and [12].

2. Proof of main results

We study the nonnegative solutions of Eq. (10) with (4) by considering Eq. (10) subject to the initial conditions

$$g(\xi) = \alpha, \qquad g'(\xi) = C, \tag{11}$$

where $C, \xi < 0$ and $\alpha > 0$ are parameters. In what follows, the solution of the initial value problem (10), (11) is denoted by $g(x)$, $g(x, \alpha)$, or $g(x, \xi, \alpha, C)$ depending on the context. Note that the function $h(x)/g^{1/N}$ is locally Lipschitz for $g > 0$ and for any x. Thus, the classical existence and uniqueness theorem of the initial value problem is applicable. Note also that $h(x)$ has only one zero point at the origin and that $h(x) < 0$ for $x < 0$ and $h(x) > 0$ for $x > 0$. Throughout the paper, we always assume $N \in (0, 1]$, $\xi < 0$ and $C < 0$ unless stated otherwise.

To begin with, we assume that ξ is fixed. For any given $\alpha > 0$ and any C, the unique solution $g(x)$ of the initial value problem (10), (11) exists on an interval containing ξ. We then extend the interval toward the right hand side until it blows-up, i.e., until $g(y) = 0$ where $y > \xi$. We define the right-maximal interval of the solution $g(x)$ of (10), (11) by $[\xi, y)$. This means that if $y < \infty$, then y is the largest value of x such that $g(x)$ exists on $[\xi, y]$ and the equation is satisfied for $x \in [\xi, y)$. In the paper, this value $y = y(\alpha) = y(\alpha, \xi, C)$ is called the vanishing point of $g(x)$.

Our goal is to prove that for any $\xi < 0$ there exists an interval $(C_1, C_2) \subset (-\infty, 0)$, depending on ξ, such that for each $C \in (C_1, C_2)$ there exist at least two values of α, $\alpha_1 \neq \alpha_2$, such that solutions $g(x, \xi, \alpha_1, C)$ and $g(x, \xi, \alpha_2, C)$ of (10), (11) satisfy $g(1, \alpha_1) = g(1, \alpha_2) = 0$.