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Dimension of divergence sets for the pointwise convergence of the Schrödinger equation ☆

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ABSTRACT

In this note, we consider the pointwise convergence along curves for the Schrödinger equation and obtain estimates for the capacitary dimension of divergence sets which extend our previous result in [6].

1. Introduction

Let us consider the initial value problem for the free Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = 0, & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = f(x) \in H^s(\mathbb{R}^d), \end{cases}$$

where H^s is the L^2-Sobolev space of order s of which norm is defined by $\|f\|_{H^s}^2 = \int_{\mathbb{R}^2} (1+|\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi$. Then, the solution is formally written as

$$u(x,t) = e^{it\Delta} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^2)} \widehat{f}(\xi) \, d\xi.$$

The problem of determining the optimal s which guarantees

$$\lim_{t\to 0} u(x,t) = f(x) \quad \text{a.e.}$$

whenever $f \in H^s$ was first consider by Carleson [5]. When spatial dimension $d = 1$, the convergence is true if and only if $s \geqslant \frac{1}{4}$ (see Carleson [5], and Dahlberg and Kenig [7]). In higher dimension, the convergence for $s > 1/2$ was shown by Sjölin [14] and Vega [17], independently. Further progress was made in connect with Fourier restriction estimates for the paraboloid [2,4,8,11,15,16]. In fact, the best known regularity for $d = 2$ is $s > \frac{3}{8}$ [8]. Recently, when $d \geqslant 3$, Bourgain [3] obtained the convergence holds for $s > \frac{1}{2} - \frac{1}{4d}$ and the necessary condition $s \geqslant \frac{1}{2} - \frac{1}{d}$.

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The sets of Lebesgue measure zero are still large in some sense. So, one may consider to refine the problem. A natural extension is to in ask the dimension of divergence set. For $f \in H^s$, let us define the divergence set to be

$$\{x \in \mathbb{R}^d\colon\ u(x,t) \text{ does not converge to } f(x) \text{ as } t \to 0\}.$$

In this direction, Sjölin and Sjoğren [12] obtained some bound of the Hausdorff dimension of the divergence set. Recently, Barceló, Bennett, Carbery and Rogers [1] obtained strengthened results.

The aforementioned problem can be thought of as convergence to the initial datum along the vertical lines (x,t), $t \in \mathbb{R}$. A natural generalization is the convergence along curves. In [6] relation between tangency of the curves and required regularity for almost everywhere convergence was investigated and a sharp result was obtained when $d = 1$. In this short note we extend the result in [6] by measuring the dimension of divergence set. For this purpose, we use the capacitary dimension.

Capacitary dimension

Let $0 < \alpha \leqslant 1$. We say that a positive Borel measure μ is α-dimensional if

$$\mu\big(I(x,r)\big) \leqslant cr^{\alpha}, \quad (x,r) \in \mathbb{R} \times \mathbb{R}+. \tag{1}$$

Here, $I(x,r)$ is the interval centered at x with radius r. Let A be a subset in $\mathbb{R}^d$ and denote by $\mathcal{M}^{\alpha}(A)$ the set of α-dimensional measures which have compact support contained in A with $0 < \mu(A) < \infty$. Then the capacitary dimension is defined by

$$\dim_c A = \sup\{\alpha\colon\ \exists \mu \in \mathcal{M}^{\alpha}(A)\}.$$

If such α does not exist, $\dim_c A = 0$. Of course, this occurs only if $A = \emptyset$. By Frostman's lemma, it follows that there exists an α-dimensional probability measure μ on A if and only if Hausdorff dimension of $A \geqslant \alpha$ provided that A is a Borel set [10]. In this case, the capacitary dimension of A equals the Hausdorff dimension.

Convergence along variable curves in $\mathbb{R} \times \mathbb{R}$

Let γ be a continuous function such that

$$\gamma : \mathbb{R} \times [-1,1] \to \mathbb{R}, \quad \gamma(x,0) = x.$$

We consider the class of curves which satisfy the followings:

- Hölder condition of order κ, $0 < \kappa \leqslant 1$,

$$\big|\gamma(x,s) - \gamma(x,t)\big| \leqslant C_0|s-t|^{\kappa}, \quad \forall s,t \in [0,1]. \tag{2}$$

- Bilipschitz condition

$$C_1|x-y| \leqslant \big|\gamma(x,t) - \gamma(y,t)\big| \leqslant C_2|x-y|, \quad \forall x,y. \tag{3}$$

Here the parameter κ may be considered to represent the degree of tangential convergence and the most typical example is $\gamma(x,t) = x - t^{\kappa}$.

Now we define for $f \in H^s$

$$\mathfrak{D}(\gamma, f) = \{x\colon\ e^{it\Delta} f\big(\gamma(x,t)\big) \text{ does not converge to } f(x) \text{ as } t \to 0\}.$$

In [6] convergence depending on κ was studied and it was shown that the divergence set $\mathfrak{D}(\gamma, f)$ has Lebesgue measure zero whenever $f \in H^s$ with $s > \max\{\frac{1}{4}, \frac{1-2\kappa}{2}\}$. The following is our main result.

Theorem 1.1. *Let $0 < \kappa \leqslant 1$. Suppose that (2) and (3) hold. If $f \in H^s$ with $s > \frac{1}{4}$, then*

$$\dim_c\big(\mathfrak{D}(\gamma, f)\big) \leqslant \max\left\{1 - 2s, \frac{1-2s}{2\kappa}\right\}.$$

Compared to the result in [1] which deals with convergence along vertical line and corresponds to the case $\kappa = 1$ this extends the convergence to the curves which approaches tangentially to initial data. Theorem 1.1 is a straightforward consequence of the following.