J. Math. Anal. Appl. 411 (2014) 271–280

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Large deviations for heavy-tailed random elements in convex cones

Christoph Kopp [1], Ilya Molchanov *

Institute of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

ARTICLE INFO

Article history:
Received 31 May 2013
Available online 25 September 2013
Submitted by U. Stadtmueller

Keywords:
Large deviation
Convex cone
Heavy-tailed distribution
Embedding
Random set

ABSTRACT

We prove large deviation results for sums of heavy-tailed random elements in rather general convex cones being semigroups equipped with a rescaling operation by positive real numbers. In difference to previous results for the cone of convex sets, our technique does not use the embedding of cones in linear spaces. Examples include the cone of convex sets with the Minkowski addition, positive half-line with maximum operation and the family of square integrable functions with arithmetic addition and argument rescaling.

1. Introduction

Most results concerning limiting behavior of sums of random elements in linear spaces can be extended for random closed sets in linear spaces, see [10, Ch. 3]. The sum of sets is defined in the Minkowski sense, i.e. the sum of two sets is the closure of the set of pairwise sums of elements from these sets. It is well known that this addition is not invertible. The most typical way to handle this setting is to consider first random convex compact sets and embed them into the Banach space of continuous functions on the unit sphere in the dual space using the support function. Then the Minkowski sum of sets corresponds to the arithmetic sum of their support functions and the Hausdorff distance between sets turns into the uniform distance in the space of support functions, which opens the possibility to use the results available for random elements in Banach spaces, see e.g. [4]. Finally, it is usually argued that the results for possibly non-convex random compact sets are identical to their convex case counterparts in view of the convexification property of the Minkowski addition, see [1].

The family of limit theorems for random sets has been recently extended with several large deviation results in the heavy-tail setting in [8] and [9]. The crucial assumption is the regular variation condition on the tail, which is similar to one that appears in limit theorems for unions of random closed sets, see [10, Ch. 4]. Let S_n denote the Minkowski sum of i.i.d. regularly varying random compact sets $\xi_1, \dots, \xi_n$ in $\mathbb{R}^m$ with tail index $\alpha > 0$ and tail measure μ. In particular, [8] show that

$$\gamma_n \mathbf{P}(S_n \in \lambda_n U) \to \mu(U) \qquad (1)$$

* Corresponding author.
E-mail addresses: chris.kopp@stat.unibe.ch (C. Kopp), ilya.molchanov@stat.unibe.ch (I. Molchanov).
[1] Partially supported by Swiss National Foundation Grant Nr. 200021-126503.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.09.042

for λ_n that grows sufficiently fast and all μ-continuous measurable subsets U of the family of all compact sets bounded away from zero. The sequence of normalizing constants $\{\gamma_n\}$ is related to the tail behavior of the norm of a single random compact set defined as its Hausdorff distance to the origin. Especially, it is required that $\lambda_n/n \to \infty$ in case $\alpha \geqslant 1$.

This large deviation result has been refined in [9], where it is shown that, for regularly varying convex random compact sets with integrable norm (so $\alpha \geqslant 1$),

$$\gamma_n \mathbf{P}(S_n \in \lambda_n U + n\mathbf{E}\xi_1) \to \mu(U), \tag{2}$$

where λ_n grows slower than in (1) and $\mathbf{E}\xi_1$ is the expectation of ξ_1, see [10, Sec. 2.1]. The method of the proof is based on the embedding argument combined with a use of classical large deviation results from [11] and [6].

The setting of random compact sets can be considered as a special case of random elements in convex cones (also called conlinear spaces), being semigroups with a scaling operation by positive reals, see [3]. A simple example is the cone of positive numbers with the maximum operation. It should be noted that in that case the embedding argument is not applicable any longer, so one has to prove the corresponding results in the cone without using any centering or symmetrization arguments.

In this paper, we generalize the above mentioned results from [8] and [9] for heavy-tailed random elements in convex cones. While the general scheme of our proofs follows the lines of the proofs from [8] and [9], it requires extra care caused by the impossibility to use the embedding device. In particular, this concerns our generalization of (2), since there is no generally consistent definition of the expectation in convex cones.

2. Regularly varying random elements in cones

A Borel function $f : (c, \infty) \mapsto (0, \infty)$ for some $c > 0$ is said to be *regularly varying* (at infinity) with index ρ if

$$f(\lambda x)/f(x) \to \lambda^\rho \quad \text{as } x \to \infty$$

for all $\lambda > 0$, see e.g. [2]. If $\rho = 0$, then f is called *slowly varying* and usually denoted by the letter ℓ instead of f. Any regularly varying function f with index ρ has a representation $f(x) = x^\rho \ell(x)$ for a slowly varying function ℓ. We write $f \sim g$ as a shorthand for $f(x)/g(x) \to 1$ as $x \to \infty$.

Theorem 2.1 *(Karamata). (See Th. 1.5.11 [2].) If f is regularly varying with index ρ and locally bounded on $[a, \infty)$, then*

(i) *for any $\beta \geqslant -(\rho + 1)$,*

$$\lim_{x\to\infty} \frac{x^{\beta+1} f(x)}{\int_a^x t^\beta f(t)\,\mathrm{d}t} = \beta + \rho + 1;$$

(ii) *for any $\beta < -(\rho + 1)$ (and for $\beta = -(\rho + 1)$ if $\int_a^\infty t^{-(\rho+1)} f(t)\,\mathrm{d}t < \infty$),*

$$\lim_{x\to\infty} \frac{x^{\beta+1} f(x)}{\int_x^\infty t^\beta f(t)\,\mathrm{d}t} = -(\beta + \rho + 1).$$

Below we summarize several concepts from [3] concerning general convex cones. A *convex cone* $\mathbb{K}$ is a topological semigroup with neutral element $\mathbf{e}$ and an extra operation $x \mapsto ax$ of scaling $x \in \mathbb{K}$ by a positive number a, so that $a(x + y) = ax + ay$ for all $a > 0$, $x, y \in \mathbb{K}$. It should be noted that we do *not* require the validity of the second distributivity law $(a + b)x = ax + bx$. The second distributivity law holds for the cone of compact sets in $\mathbb{R}^d$ with the Minkowski addition and enables using the embedding argument.

We assume that $\mathbb{K}$ is a pointed cone, i.e. ax converges to the cone element $\mathbf{0}$ called the origin as $a \downarrow 0$ for all $x \neq \mathbf{e}$. Assume that $\mathbb{K}$ is metrized by a homogeneous metric d, i.e. $d(ax, ay) = ad(x, y)$ for all $x, y \in \mathbb{K}$ and $a > 0$. The value $\|x\| = d(x, \mathbf{0})$ is called the *norm* of x which in general constitutes an abuse of language since $\|\cdot\|$ is not necessarily sub-linear. Nevertheless, the norm is sub-linear if the metric is *sub-invariant*, i.e. if $d(x+h, x) \leqslant d(h, \mathbf{0}) = \|h\|$ for all $x, h \in \mathbb{K}$. A stronger assumption is the translation-invariance of the metric meaning that $d(x + h, y + h) = d(x, y)$ for all $x, y, h \in \mathbb{K}$. In a cone with sub-invariant metric, $\mathbf{0} = \mathbf{e}$, see [3, Lemma 2.7].

Furthermore, $\mathbb{S} = \{x \in \mathbb{K}\colon\ \|x\| = 1\}$ denotes the unit sphere. For $\varepsilon > 0$,

$$A^\varepsilon = \{x \in \mathbb{K}\colon\ d(x, A) \leqslant \varepsilon\}$$

is the ε-envelope of $A \subset \mathbb{K}$, where $d(x, A) = \inf_{a\in A} d(x, a)$. The Borel σ-algebra on $\mathbb{K}$ is denoted by $\mathcal{B}$ and used to define random cone elements ξ as measurable maps from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to $(\mathbb{K}, \mathcal{B})$.

Furthermore, $\operatorname{int} A$, $\operatorname{cl} A$ and ∂A denote the interior, closure and boundary of $A \subset \mathbb{K}$. A set $A \subset \mathbb{K}$ is said to be bounded away from a point $x \in \mathbb{K}$ if $x \notin \operatorname{cl} A$. If μ is a measure on $\mathcal{B}$, then $A \in \mathcal{B}$ is called a μ-continuity set if $\mu(\partial A) = 0$.