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ZL-amenability and characters for the restricted direct products of finite groups

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ABSTRACT

Let G be a restricted direct product of finite groups $\{G_i\}_{i\in I}$, and let $Z\ell^1(G)$ denote the centre of its group algebra. We show that $Z\ell^1(G)$ is amenable if and only if G_i is abelian for all but finitely many i, and characterize the maximal ideals of $Z\ell^1(G)$ which have bounded approximate identities. We also study when an algebra character of $Z\ell^1(G)$ belongs to c_0 or ℓ^p and provide a variety of examples.

1. Introduction

The L^1-convolution algebra of a locally compact group G is amenable if and only if the group G is amenable. In contrast, there is as yet no 'intrinsic' characterization of those groups G for which $ZL^1(G)$, the centre of $L^1(G)$, is amenable. It will be convenient to call such groups *ZL-amenable*.

The article [1] studied the problem of which compact groups are ZL-amenable. One can consider the corresponding problem for (discrete) FC groups, that is, the groups in which each conjugacy class is finite. We note that the problem is solved for *finitely generated* FC groups: for, by a result of B.H. Neumann [13, Theorem 5.1], each such group has a finite derived subgroup; and when a discrete group has a finite derived subgroup, then it is ZL-amenable by a result of Stegmeir [17, Theorem 1].

The present paper is concerned with a particular class of infinitely generated FC groups, namely the restricted direct products of finite groups (*RDPF groups*, for short) whose formal definition is given below in Definition 3.1. The C* and von Neumann algebras of discrete RDPF groups were studied in some old work of Mautner [11], but to our knowledge the centres of their ℓ^1-group algebras have not been studied explicitly.

When G is such a group, we are able to study $Z\ell^1(G)$ in some detail. We show in Theorem 3.5 that an RDPF group is ZL-amenable if and only if its derived subgroup is finite. Then, in Theorem 4.3, we obtain a characterization of those RDPF groups G for which every maximal ideal in $Z\ell^1(G)$ has a bounded approximate identity. (For a commutative, unital Banach algebra, the condition that each maximal ideal has a bounded approximate identity can be thought of as a weaker version of amenability. It has been rediscovered and studied in further depth under the name of *character amenability*, see e.g. [8,12] for further details, not restricted to the unital or commutative cases.)

Using our techniques, in Section 5, we revisit the main counter-example from Stegmeir's paper [17], and are able to give simpler proofs of some of his results. In doing so, we are led to consider when given characters on $Z\ell^1(G)$ belong to c_0,

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when viewed as functions on G in the natural way. We obtain a necessary and sufficient condition for this to occur, and also study the related question of when such characters lie in ℓ^p for various exponents $p \in [1, \infty)$.

2. Notation and other preliminaries

In this section we set up some notation and some preliminary results that will be needed. Most of these results are well-known to specialists in non-abelian harmonic analysis but we have repeated them, often in special cases, to keep the present paper more self-contained.

2.1. Important conventions

Since we are interested in discrete groups, we will always equip finite groups with counting measure, not the uniform probability measure. Occasionally this will mean that in quoting results concerning harmonic analysis on compact groups, we have to insert a scaling factor. It should be clear, from context, when and how this is done.

Infinite products and sums over arbitrary indexing sets. In several places we will want to consider certain infinite sums or products where the numbers involved are indexed by some arbitrary set $\mathbf{I}$. Since we only need to consider sums of positive numbers and products of numbers $\geqslant 1$, we shall interpret this as unconditional summation

$$\sum_{i\in\mathbf{I}} a_i := \sup\left\{\sum_{i\in F} a_i:\ F \subseteq \mathbf{I},\ |F| < \infty\right\}$$

when $a_i \geqslant 0$ for all $i \in \mathbf{I}$; and likewise we define

$$\prod_{i\in\mathbf{I}} b_i := \sup\left\{\prod_{i\in F} b_i:\ F \subseteq \mathbf{I},\ |F| < \infty\right\}$$

when $b_i \geqslant 1$ for all i.

Algebra characters and group characters. The Gelfand spectrum of a commutative, unital Banach algebra A will be denoted by $\mathrm{sp}(A)$; usually we shall think of it as the space of non-zero multiplicative linear functionals, rather than the maximal ideal space.

Unfortunately, the word ‘character’ is used in both Banach algebra theory and in the representation theory of finite groups, and means two slightly different things. In this article, to try and prevent ambiguity, we will always use the phrase *algebra character* to mean a character in the sense of Gelfand theory, i.e. a non-zero multiplicative linear function from a complex Banach algebra to the ground field; and we will always use the phrase *group character* to mean a character in the sense of finite group theory, i.e. the trace of a finite-dimensional (unitary) representation.

Given a group character χ, we say that π *affords* χ when π is a finite-dimensional representation of G whose trace is χ. The *degree of* χ, which we denote by d_χ, is defined to be the dimension of any π which affords χ; equivalently, $d_\chi = \chi(e)$ where e is the identity of the group in question.

Remark 2.1. When viewing a group character as the trace of a suitable representation, we use the unnormalized trace, denoted by Tr, so that the trace of the $d \times d$ identity matrix is d. (In some sources, one normalizes group characters so that they each take the value 1 on the identity element of the group; our convention is more in keeping with that used in finite group theory.)

Notation. Recall that if G is a (discrete) group, its *derived subgroup* or *commutator subgroup* is the normal subgroup of G generated by the set of all commutators of elements in G. We shall denote the derived subgroup of G by G'.

2.2. General properties of $\mathrm{Z}\ell^1$

It is well known that when G is finite, algebra characters on $\mathrm{Z}\ell^1(G)$ correspond to irreducible group characters of G. More precisely, we have the following result.

Lemma 2.2. *Let G be a finite group. If ψ is an algebra character on $\mathrm{Z}\ell^1(G)$, then there is a unique irreducible group character χ of G which satisfies*

$$\psi(f) = \sum_{x\in G} f(x) {d_\chi}^{-1} \chi\left(x^{-1}\right) \quad \text{for all } f \in \mathrm{Z}\ell^1(G). \tag{2.1}$$

Conversely, for each irreducible group character χ on G, the formula (2.1) defines an algebra character on $\mathrm{Z}\ell^1(G)$.

This result is well known and has been generalized to the setting of locally compact $[\mathrm{FC}]^-$ groups (in particular, it is a very special case of [10, Corollary 1.3]). Nevertheless, we shall give a self-contained outline of the proof of Lemma 2.2, since this is an instructive warm-up for the arguments of Section 4, and may help to make that section more accessible.