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Continuous random walks and fractional powers of operators

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ABSTRACT

We derive a probabilistic representation for the Fourier symbols of the generators of some stable processes. This short paper represents a bridge between probabilists and researchers working in PDE's.

1. Introduction

The connection between fractional operator in space and diffusion with long jumps has been pointed out by many researchers (see for example [1,13,19] and the references therein). It is well known that the compound Poisson process is a continuous time stochastic process with jumps which arrive, according to a Poisson process, with specific probability law for the size. Our aim is to characterize the jumps distribution in order to obtain singular limit measure characterizing fractional powers of operators.

2. Preliminaries

Let $N(t)$, $t > 0$ be a Poisson process with rate $\lambda > 0$. Let Y_j, $0 \leqslant j \leqslant n$ be $n+1$ independent and identically distributed (i.i.d.) random jumps such that $Y_j \sim Y$ for all j, where the symbol "$\sim$" stands for equality in law. That is, $X_1 \sim X_2$ if, for every Borel set $\mathcal{B}$, $P\{X_1 \in \mathcal{B}\} = P\{X_2 \in \mathcal{B}\}$ and therefore the random variables X_1 and X_2 have the same distribution or probability function. We note that, if probabilities are defined for a larger class of events, it is possible for two random variables to have the same distribution function but not the same probability for every event (see [5], Chapter 2). Furthermore, two random variables which are identically distributed are not necessarily equal.

It is well known that

$$Z_t = \sum_{j=0}^{N(t)} Y_j - \lambda t \mathbb{E}Y, \quad t > 0 \tag{2.1}$$

($\mathbb{E}$ is the mean operator and $\mathbb{E}Y = \int y P\{Y \in dy\}$) is the compensated compound Poisson process with generator

$$(\mathcal{A}f)(x) = \lambda \int_{\mathbb{R}} \big(f(x+y) - f(x) - yf'(x)\big)\nu_Y(dy) \tag{2.2}$$

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where $\nu_Y : \Omega \subseteq \mathbb{R} \mapsto [0,1]$ is the density law of $Y \in \Omega$. The process (2.1) is a compensated process involving the compound Poisson process $\sum_{j=0}^{N(t)} Y_j$. Thus, we have a sequence of i.i.d. random jumps with common law ν_Y and a random number of jumps $1 + N(t)$ with $P\{N(t) = n\} = e^{-\lambda t}(\lambda t)^n/n!$, $n = 0, 1, 2, \dots$, $t > 0$. It is worth to mention that [17,18] the transport equation

$$\frac{\partial u}{\partial t} = \mathcal{A}u - \lambda u + \lambda K u = \mathcal{A}u - \lambda(I - K)u \tag{2.3}$$

where

$$\mathcal{A}u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\big(a(x)u\big)$$

(assume that a is sufficiently smooth) and K is the Frobenius–Perron operator corresponding to the transformation $T(x) = x - \tau(x)$ has a solution which is the law of the solution to the Poisson driven stochastic differential equation

$$d\mathbf{X}_t = a(\mathbf{X}_t)\,dt + \tau(\mathbf{X}_t)\,d\mathbf{N}_t. \tag{2.4}$$

Formula (2.4) says that $\mathbf{X}_t$ is a continuous time stochastic process with jumps τ which arrive randomly according to the Poisson process $\mathbf{N}_t$. In particular, $d\mathbf{N}_t = 1$ if a Poisson event arrives or $d\mathbf{N}_t = 0$ otherwise. The compensated Poisson process $Z_t = N(t) - \lambda t$ (take $Y_j \equiv 1$ for all j in formula (2.1)) is therefore governed by the equation

$$\frac{\partial u}{\partial t}(x,t) = \lambda \frac{\partial u}{\partial x}(x,t) - \lambda(I-K)u(x,t) = \lambda \frac{\partial u}{\partial x}(x,t) - \lambda\big(u(x,t) - u(x-1,t)\big)$$

where the Frobenius–Perron operator is associated to the jump function $\tau \equiv 1$. The first derivative $\partial u/\partial x$ disappears if $Z_t = N(t)$ is the Poisson process. Eq. (2.3) appears in such diverse areas as population dynamics (see for example [11,14]) and in astrophysics [4].

Formula (2.2) is quite familiar in the representation of the fractional power of the Laplacian. Indeed, the fractional Laplace operator can be defined pointwise:

$$-(-\triangle)^\alpha f(\mathbf{x}) = \int_{\mathbb{R}^d} \big(f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - \mathbf{y}\cdot\nabla f(\mathbf{x})\mathbf{1}_{(|\mathbf{y}|\leqslant 1)}\big)\frac{C_d(\alpha)\mathbf{dy}}{|\mathbf{y}|^{2\alpha+d}} \tag{2.5}$$

where $C_d(\alpha)$ is a constant depending on d and $\alpha \in (0,1)$, f is a suitable test function, C^2 function with bounded second derivative for instance. We note that ν_Y in (2.2) is a density defining a probability measure. Furthermore, the fractional Laplacian is the governing operator of symmetric stable processes.

Let us introduce the 1-dimensional β-stable process $\mathfrak{S}^\beta(t)$, $t > 0$ with no drift. In this case we have the characteristic function $\mathbb{E}\exp i\xi\mathfrak{S}^\beta(t) = \exp -t\Psi(\xi)$ with Fourier symbol

$$\Psi(\xi) = |\xi|^\beta \exp\left(-i\frac{\pi\gamma}{2}\frac{\xi}{|\xi|}\right) = \sigma|\xi|^\beta\left(1 - i\theta\frac{\xi}{|\xi|}\tan\frac{\pi\beta}{2}\right), \quad \beta \in (0,1)\cup(1,2]$$

where $\sigma = \cos\pi\gamma/2$, $\theta = \cot(\frac{\pi\beta}{2})\tan(\frac{\pi\gamma}{2})$ and γ must be determined in such a way that $\theta \in [-1,1]$ and $\sigma > 0$. The parameter θ is the skewness parameter. In particular, $\mathfrak{S}^\beta(t)$ is a symmetric real-valued stable process for $\theta = 0$ (and $\beta \in (0,1)\cup(1,2]$). If $\beta \in (0,1)$ and $\theta = -1$ then $\mathfrak{S}^\beta(t)$ is totally negatively skewed whereas, if $\beta \in (0,1)$ and $\theta = 1$ (that is $\gamma = \beta$) then $\mathfrak{S}^\beta(t)$ is totally positively skewed. In the latter case, the stable process is also termed stable subordinator (see for example [2]) and we will denote such a process by $\mathfrak{H}^\alpha(t)$, $t > 0$, $\alpha \in (0,1)$. We recall that a random variable is positively skewed if the right tail of its probability distribution is longer than the left tail. The converse holds for the negative case.

Compound Poisson and stable processes belong to the general class of Lévy processes whose characteristic function is written in terms of the following Fourier symbol (Lévy–Khintchine)

$$\Psi_L(\boldsymbol{\xi}) = i\mathbf{b}\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot M\boldsymbol{\xi} - \int_{\mathbb{R}^d-\{0\}} \big(e^{i\boldsymbol{\xi}\cdot\mathbf{y}} - 1 - i\boldsymbol{\xi}\cdot\mathbf{y}\mathbf{1}_{(|\mathbf{y}|\leqslant 1)}\big)\mu(d\mathbf{y}) \tag{2.6}$$

where $\mathbf{b} \in \mathbb{R}^d$, M is a positive definite symmetric $d \times d$ matrix and μ is a Lévy measure on $\mathbb{R}^d - \{0\}$, that is a Borel measure on $\mathbb{R}^d - \{0\}$ such that

$$\int \big(|\mathbf{y}|^2 \wedge 1\big)\mu(d\mathbf{y}) < \infty \quad \text{or equivalently} \quad \int \frac{|\mathbf{y}|^2}{1+|\mathbf{y}|^2}\mu(d\mathbf{y}) < \infty. \tag{2.7}$$

If D_t, $t > 0$ is a subordinator (not necessarily stable), then its Lévy symbol is written as

$$\eta(\xi) = ib\xi + \int_0^\infty \big(e^{i\xi y} - 1\big)\mu(dy) \tag{2.8}$$