J. Math. Anal. Appl. 411 (2014) 372–380

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compact covers and function spaces ☆

J. Kąkol [a], M. Lopez-Pellicer [b,*], O. Okunev [c]

[a] Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznań, Poland
[b] Depto. de Matemática Aplicada and IUMPA, Universitat Politècnica de València, E-46022 València, Spain
[c] Facultad de Ciencias Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, Av. San Claudio y Rio Verde s/n, Col. San Manuel, Ciudad Universitaria, CP 72570 Puebla, Puebla, Mexico

ARTICLE INFO

Article history:
Received 4 April 2013
Available online 27 September 2013
Submitted by B. Cascales

Keywords:
Compact resolution
$C(X)$- and $L_p(X)$-spaces
Čech-complete space
Lindelöf Σ, K-analytic, analytic spaces
ℓ_p-, ℓ_c- and t-equivalence
Pointwise countable type spaces
Polish space
Real compactification
μ-space
Web-bounded spaces

ABSTRACT

For a Tychonoff space X, we denote by $C_p(X)$ and $C_c(X)$ the space of continuous real-valued functions on X equipped with the topology of pointwise convergence and the compact-open topology respectively. Providing a characterization of the Lindelöf Σ-property of X in terms of $C_p(X)$, we extend Okunev's results by showing that if there exists a surjection from $C_p(X)$ onto $C_p(Y)$ (resp. from $L_p(X)$ onto $L_p(Y)$) that takes bounded sequences to bounded sequences, then υY is a Lindelöf Σ-space (respectively K-analytic) if υX has this property. In the second part, applying Christensen's theorem, we extend Pelant's result by proving that if X is a separable completely metrizable space and Y is first countable, and there is a quotient linear map from $C_c(X)$ onto $C_c(Y)$, then Y is a separable completely metrizable space. We study also a non-separable case, and consider a different approach to the result of J. Baars, J. de Groot, J. Pelant and V. Valov, which is based on the combination of two facts: Complete metrizability is preserved by ℓ_p-equivalence in the class of metric spaces (J. Baars, J. de Groot, J. Pelant). If X is completely metrizable and ℓ_p-equivalent to a first-countable Y, then Y is metrizable (V. Valov). Some additional results are presented.

1. Introduction

All spaces considered in this article are assumed to be completely regular and Hausdorff. We use terminology and notation as in [10]. We say that a set A in a space X is *functionally bounded in* X if every continuous real-valued function on X is bounded on A. A space X is a μ*-space* if every closed functionally bounded subspace of X is compact. A *Polish space* is a separable completely metrizable space. The symbol ω denotes the smallest infinite ordinal (so ω is the set of all non-negative integers).

We denote by $C_p(X)$ and $C_c(X)$ the spaces of continuous real-valued functions on X endowed with the topology of pointwise convergence and the compact-open topology respectively. $L_p(X)$ is the topological dual of $C_p(X)$ endowed with the weak* topology. We assume that X is a subspace of $L_p(X)$ by virtue of the standard embedding $x \mapsto \hat{x}$ where $\hat{x}(f) = f(x)$ for each $f \in C_p(X)$.

Recall that Nagata's theorem states that if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then X and Y are homeomorphic, see [5]. This suggests the following problem, see [2].

☆ The research was supported for the first named author by National Center of Science, Poland, Grant No. N N201 605340, and for the first and second named authors by Generalitat Valenciana, Conselleria d'Educació i Esport, Spain, Grant PROMETEO/2013/058.

* Corresponding author.
E-mail addresses: kakol@amu.edu.pl (J. Kąkol), mlopezpe@mat.upv.es (M. Lopez-Pellicer), oleg@fcfm.buap.mx (O. Okunev).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.09.046

Two spaces X and Y are said to be *t-equivalent* (*ℓ_p-equivalent*) if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic). We say that a topological property $\mathcal{P}$ is preserved by t-equivalence (ℓ_p-equivalence) if whenever two spaces X and Y are t-equivalent (ℓ_p-equivalent) and X has the property $\mathcal{P}$, Y has the property $\mathcal{P}$ too.

What topological properties are preserved by the relations of t-equivalence and ℓ_p-equivalence?

Clearly, a property $\mathcal{P}$ is preserved by the relation of t-equivalence (ℓ_P-equivalence) if and only if there is a *dual* topological (linear topological) property $\mathcal{Q}$ such that X has $\mathcal{P}$ if and only if $C_p(X)$ has $\mathcal{Q}$; in different words, if $\mathcal{P}$ "admits a description in terms of the (linear) topological structure of $C_p(X)$".

We will say that two spaces X and Y are *ℓ_c-equivalent* if the spaces $C_c(X)$ and $C_c(Y)$ are linearly homeomorphic. Note that if X and Y are ℓ_p-equivalent and X is a μ-space, then the spaces X and Y are also ℓ_c-equivalent, see [4]. It is well known (by a combination of Milyutin's and Pestov's results, see [4, Theorem 3]) that $[0,1]$ and $[0,1]\times[0,1]$ are ℓ_c-equivalent but not ℓ_p-equivalent. On the other hand, if X and Y are ℓ_p-equivalent and X is *Dieudonné complete* (in particular, if X is paracompact or realcompact), then X and Y are ℓ_c-equivalent, see [4, Theorem 1].

We say that a space X *ℓ_c-covers* a space Y if there is a continuous open linear mapping from $C_c(X)$ onto $C_c(Y)$. Clearly, if X and Y are ℓ_c-equivalent, then each of the two ℓ_c-covers the other.

There are many known results about preservation and non-preservation of various topological properties by t-equivalence and ℓ_p-equivalence; see, e.g., [4–6,20,27]. For example, metrizability, local compactness, the countability of weight, normality and paracompactness are not ℓ_p-invariant. On the other hand, hemicompactness, the property of being an $\aleph_0$-space, the Lindelöf Σ-property, K-analyticity and analyticity are preserved by ℓ_p-equivalence.

We denote by $\mathbb{P}$ the space ω^ω endowed with the Tychonoff product topology (with all the factors discrete). We equip the space $\mathbb{P}$ with the natural partial order: $p\leqslant q$ if and only if $p(n)\leqslant q(n)$ for all $n\in\omega$. For an element p of $\mathbb{P}$ and a natural number k we denote by $p|k$ the finite sequence $(p(1),\dots,p(k))$. Given a finite sequence $\sigma=(\sigma_1,\dots,\sigma_n)$ of natural numbers, we denote by $W(\sigma)$ the set $\{p\in\mathbb{P}\colon\ p|n=\sigma\}$. Clearly, for every $p\in\mathbb{P}$, the family of sets $\{W(p|n)\colon\ n\in\omega\}$ is a base of open neighborhoods of p in $\mathbb{P}$.

A family of subspaces $\mathcal{R}=\{A_p\colon\ p\in\mathbb{P}\}$ of a space X is called a *resolution* of X if it covers X and $A_p\subset A_q$ whenever $p\leqslant q$. We say that a resolution $\mathcal{R}$ is *compact* if each element A_p of $\mathcal{R}$ is compact. If X is a topological vector space, we say that a resolution $\mathcal{R}$ of X is *bounded* if each element of $\mathcal{R}$ is bounded in X (that is, absorbed by any neighborhood of zero). A resolution $\mathcal{R}$ *swallows compact sets* if every compact subspace of X is contained in some element of $\mathcal{R}$.

As usual, a set-valued mapping $T\colon X\to Y$ is called *compact valued* if the set $T(x)$ is compact for every $x\in X$, and is *upper semicontinuous* if for every open set V in Y, the set $\{x\in X\colon\ T(x)\subset V\}$ is open. We abbreviate "compact valued upper semicontinuous" as "usco". For a set $A\subset X$ we denote $T(A)=\bigcup\{T(x)\colon\ x\in A\}$, and we say that T is *onto* Y if $T(X)=Y$. We denote the family of all compact subspaces of a space X by $\mathcal{K}(X)$ (so compact-valued mappings to X are the same as functions to $\mathcal{K}(X)$).

In Section 2 we find a characterization of Lindelöf Σ-property of υX in terms of the linear topological structure of $C_p(X)$ and use it to show that if υX is a Lindelöf Σ-space or a K-analytic space and there exists a surjection from $L_p(X)$ onto $L_p(Y)$ that takes bounded sequences to bounded sequences, then υY is a Lindelöf Σ-space (respectively, K-analytic); we prove a similar statement for the Lindelöf Σ-property of υX and mappings between $C_p(X)$ and $C_p(Y)$. This supplements some earlier results of Okunev [18].

A. Arhangel'skii asks in [3, Problem 20] if *a first-countable space Y which is ℓ_p-equivalent to a metrizable space X must also be metrizable*. In [6, Theorem 3.3] J. Baars, J. de Groot and J. Pelant proved that *complete metrizability is preserved by ℓ_p-equivalence in the class of metrizable spaces*. They also gave an alternative proof for separable metrizable spaces by using Christensen's theorem (Theorem 1 below), [6, Theorem 5.1, Theorem 5.3]. Later on, Valov proved, using the results in [25], that the answer to Arhangel'skii's problem is positive for Čech-complete spaces Y, see [27, Corollary 4.6]. The combination of the above facts yields: *The property of being a completely metrizable space is preserved by the ℓ_p-equivalence for spaces satisfying the first axiom of countability*.

In Section 3 of this article we discuss the preservation of complete metrizability by ℓ_c-equivalence. We give a different proof of the preservation of complete metrizability in the case of ℓ_c-equivalent spaces X and Y where X is Polish and Y is first countable, and discuss the non-separable case. Note however that from Valov's quite technical [27, Corollary 4.6] it follows that if X is completely metrizable, Y is paracompact first countable, and there is a continuous surjection from $C_c(X)$ onto $C_c(Y)$, then Y is completely metrizable. Our approach is different and uses a property of $C_c(X)$ which is preserved by linear open maps and characterizes spaces X with a compact resolution swallowing compact sets (Theorem 14, Corollary 15). The importance of this concept stems from the following deep result of J.P.R. Christensen [8, Theorem 3.3].

Theorem 1. *A metrizable space X is Polish if and only if X admits a compact resolution swallowing compact sets.*

We partially extend Theorem 1 to the non-separable case, see Proposition 16 below, and we apply this extension to show that if a space Y is ℓ_c-covered by a completely metrizable space of weight κ, then Y admits a compact cover swallowing compact sets similar to a resolution. In particular, if Y is of pointwise countable type and X is Polish, then Y is separable and completely metrizable (Corollary 24). The separable case will be deduced from Theorem 21 showing that if X is an $\aleph_0$-space and there is an open continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is an $\aleph_0$-space. Indeed, the latter fact and Theorem 1 apply to prove that if for a separable completely metrizable space X there exists a quotient linear map from $C_c(X)$ onto $C_c(Y)$ and Y is first countable, then Y is separable and completely metrizable.