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Iterative roots of piecewise monotonic functions with finite nonmonotonicity height ☆

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ABSTRACT

It is known that any continuous piecewise monotonic function with nonmonotonicity height not less than 2 has no continuous iterative roots of order n greater than the number of forts of the function. In this paper, we consider the problem of iterative roots in the case that the order n is less than or equal to the number of forts. By investigating the trajectory of possible continuous roots, we give a general method to find all iterative roots of those functions with finite nonmonotonicity height.

1. Introduction

Given a nonempty set X and a positive integer $n \in \mathbb{N}$, a function $f : X \to X$ is said to be an iterative root of $F : X \to X$ of order n if

$$f^n(x) = F(x), \quad \forall x \in X, \tag{1.1}$$

where f^n denotes the nth iterate of f, i.e., $f^n(x) = f \circ f^{n-1}(x)$ and $f^0(x) \equiv x$ for any $x \in X$. The existence of iterative roots has been intensively studied for almost 200 years, starting from Ch. Babbage [1], and great advance were made to find the solutions of Eq. (1.1) (see [2–7,11–13]). Among these works, there are plentiful results on iterative roots for monotonic self-mapping on compact interval, in which the roots are defined piece by piece from a small neighborhood without fixed points to the whole domain [5–7]. However, the method is invalid without the assumption of monotonicity and thus finding iterative roots for non-monotonic mapping is treated as a difficult problem. In 1983, Jingzhong Zhang and Lu Yang [15] studied a class of non-monotonic continuous functions, called strictly piecewise monotonic functions (abbreviated as PM functions). By introducing the idea of "characteristic interval", the problem of iterative roots for PM function can be reduced to that on its characteristic interval, which becomes the monotone case. In this paper, being different from these works on searching existence conditions for iterative roots, we try to give a general method to find all roots.

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2. Preliminaries

Let $I := [a, b]$ for $a, b \in \mathbb{R}$, $a < b$, and let $F: I \to I$ be a continuous function. A point $c \in (a, b)$ is called a *fort of* F if F is strictly monotonic in no neighborhood of c. The set of forts of F is denoted by $S(F)$. A function F is said to be *piecewise monotonic* if the number of forts of F, denoted by $N(F)$ is finite. By $\mathcal{PM}(I, I)$ we denote the set of all piecewise monotonic self-mappings of I.

Lemma 2.1. *(See Lemma 2.3 in [9].) Let $p, q \in \mathbb{R}$ be such that $p < q$ and $F([a, b]) \subset [p, q]$ and let $F : [a, b] \to \mathbb{R}$ and $G : [p, q] \to \mathbb{R}$ be continuous functions. Then*

$$S(G \circ F) = S(F) \cup \{c \in (a, b):\ F(c) \in S(G)\}.$$

In particular, the function $G \circ F$ is piecewise monotonic if and only if so are the functions F and $G|_{F([a,b])}$.

By Lemma 2.1, we know that every continuous iterative root of a piecewise monotonic (strictly monotonic) self-mapping is also piecewise monotonic (strictly monotonic). Moreover, for each function $F \in \mathcal{PM}(I, I)$, $N(F)$ is nondecreasing under iteration, i.e.,

$$0 = N(F^0) \leqslant N(F) \leqslant N(F^2) \leqslant \cdots \leqslant N(F^n) \leqslant \cdots.$$

Then we define the *nonmonotonicity height* (or simple height) $H(F)$ of F as the least $k \in \mathbb{N} \cup \{0\}$ satisfying $N(F^k) = N(F^{k+1})$ if such a k exists and ∞ otherwise.

Example 2.1. (See Example 2.9 in [9].) For the classical hat function $F : [0, 1] \to [0, 1]$, given by

$$F(x) = \min\{2x, 2 - 2x\},$$

one can check that $S(F^k) = \{\frac{1}{2^k}, \ldots, \frac{2^k-1}{2^k}\}$, $k \in \mathbb{N}$, and thus $H(F) = \infty$.

Example 2.2. Let $F : [0, 1] \to [0, 1]$ be defined by

$$F(x) = \begin{cases} \frac{1}{2}x, & \forall x \in [0, \frac{1}{2}), \\ -\frac{1}{2}x + \frac{1}{2}, & \forall x \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, F maps $[0, 1]$ onto $[0, 1/4]$, which implies that $H(F) = 1$.

Example 2.3. Consider the function $F : [0, 1] \to [0, 1]$, given by

$$F(x) = \begin{cases} x, & \forall x \in [0, \frac{1}{3}), \\ -x + \frac{2}{3}, & \forall x \in [\frac{1}{3}, \frac{2}{3}), \\ 2x - \frac{4}{3}, & \forall x \in [\frac{2}{3}, 1]. \end{cases}$$

We have $S(F^3) = S(F^2) = \{\frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$ and then $H(F) = 2$.

The simplest case for the nonmonotonicity height is $H(F) = 0$, which means that F is strictly monotonic. When $H(F) = 1$, the problem of iterative roots was reduced to be discussed on its characteristic interval (see [8,10,15,14]). More concretely, for every $F \in \mathcal{PM}(I, I)$ with $H(F) = 1$, there exists a sub-interval of I denoted by $K(F)$, covering the range of F such that F is strictly monotonic on it. Such a maximal sub-interval bounded by either forts or end-points, is called the *characteristic interval* of F. For instance, the characteristic interval of F given in Example 2.2 is $K(F) = [0, \frac{1}{2}]$, and F in Example 2.3 has no characteristic interval since $H(F) > 1$. When F is strictly increasing on its characteristic interval, the following results are obtained:

Theorem 2.1. *(See Theorem 4 in [14].) Let $F \in \mathcal{PM}(I, I)$ and $H(F) \leqslant 1$. Suppose that* (i) *F is strictly increasing on its characteristic interval $[a', b']$ and* (ii) *$F(x)$ on I cannot reach a' and b' unless $F(a') = a'$ or $F(b') = b'$. Then for any integer $n > 1$, F has a continuous iterative root of order n. Moreover, these conditions are necessary for integers $n > N(F) + 1$.*

In addition, when $H(F) > 1$, we also have the following nonexistence result of iterative roots for order $n > N(F)$.

Theorem 2.2. *(See Theorem 1 in [14].) Let $F \in \mathcal{PM}(I, I)$ and $H(F) > 1$. Then F has no continuous iterative roots of order n for $n > N(F)$.*