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On partial shadowing of complete pseudo-orbits

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ABSTRACT

In recent years many new definitions of shadowing, using a notion of ergodic (or average) pseudo-orbit were introduced. While, under the assumption of chain mixing, average pseudo-orbit usually can be equivalently replaced by pseudo-orbit in these definitions of shadowing, it is not completely clear when these shadowing properties (i.e. approximation of a pseudo-orbit by a real orbit on a sufficiently large set of indices) can or cannot occur. In this paper we analyze necessary and sufficient conditions for shadowing over a set with positive density (or syndetic). While we do not provide full characterization, a few relations to standard notions from topological dynamics are obtained.

1. Introduction

Recently, we can observe an increasing interest in concepts of pseudo-orbit when errors larger than δ are allowed, provided they are sparse enough. Particular examples of shadowing property where this approach is adopted are average shadowing property [5], ergodic shadowing property [8] and $\underline{d}$-shadowing property [1]. Since the definition of average pseudo-orbit allows "jumps" of pseudo-orbit over the space, usually chain mixing is a consequence of these definitions of shadowing. But then, often it is also possible to transform pseudo-orbit with "random jumps" into an ordinary pseudo-orbit (where for all n we have $d(f(x_n), x_{i+n}) < \delta$) by modification of the starting average pseudo-orbit on a set with density zero (or small density). Then if in the definition of shadowing, errors are allowed in the shadowing over a set with density zero (resp. small density), then in these definitions of shadowing, instead of average pseudo-orbits we can simply use ordinary pseudo-orbits. From another point of view, the question of a dynamical system with a given type of shadowing property, is in fact the question about the maximal set of integers that can be obtained when we compare the given pseudo-orbit with the orbit of properly chosen points in the space. In other words, we ask if the set

$$\{n\colon d\big(f^n(z), x_n\big) < \varepsilon\}$$

belongs to a specified family of subsets. This point of view on sub-shadowing is adopted in this paper where we are mainly interested in the family $\mathcal{F}_{\underline{d}}$ of sets with positive lower density and family $\mathcal{F}_s$ of syndetic sets.

In the present paper, we investigate relations between $\mathcal{F}_{\underline{d}}$-shadowing property or $\mathcal{F}_s$-shadowing property and other more standard notions such as weak mixing, minimality, positive entropy or shadowing property. We show that in some

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cases these two versions of shadowing are natural consequence of considered notions and in some others they will not be present except for the situation that the space is trivial. This completes and extends previous results of [1]. We show that $\mathcal{F}_{\underline{d}}$-shadowing property (or equivalently $\underline{d}$-shadowing) is implied by average shadowing property (see Theorem 5 and Lemma 4). Recently it was proved in [14] that the following implications are true:

$$\begin{aligned} &\text{almost specification}\\ &\quad\Longrightarrow\quad \text{asymptotic average shadowing property}\\ &\quad\Longrightarrow\quad \text{average shadowing property}. \end{aligned}$$

Hence, by the above result (and Theorem 5) $\underline{d}$-shadowing property is yet another weaker version of “Shadowing in average”. But this also implies that $\mathcal{F}_{\underline{d}}$-shadowing property should be more commonly observed in the dynamics, than other aforementioned properties.

2. Preliminaries

2.1. Furstenberg's families

A collection $\mathcal{F}$ of subsets of $\mathbb{N}$ is called *family*, if it is upward hereditary, that is:

$$A \subset B, \quad A \in \mathcal{F} \quad \Longrightarrow \quad B \in \mathcal{F}.$$

A family $\mathcal{F}$ is called a *filter* if it contains any finite intersections of its elements. Book of Akin [2] contains a comprehensive introduction to the topic of filters and (Furstenberg) families. We continue this section by bringing a minimum of necessary materials. The *dual family* of $\mathcal{F}$ is

$$\mathcal{F}^* := \{A \subset \mathbb{N} \colon \forall F \in \mathcal{F},\ A \cap F \neq \emptyset\}.$$

A set $A \subset \mathbb{N}$ is *syndetic* if it has bounded gaps, i.e. there is $k > 0$ such that $A \cap [i, i+k) \neq \emptyset$ for every $i \geqslant 0$. We denote by $\mathcal{F}_s$ the family of all syndetic sets and its dual is denoted by $\mathcal{F}_t = \mathcal{F}_s^*$. Elements of $\mathcal{F}_t$ are called *thick* sets. If $A \in \mathcal{F}_t$ then it contains arbitrary long blocks of consecutive integers. A set A is *piecewise syndetic* if it can be presented as the intersection of a thick set with a syndetic set. We denote by $\mathcal{F}_{ps}$ the family of all piecewise syndetic sets. Element of $\mathcal{F}_{ps}^*$ are said to be *thickly syndetic* sets. Note that a set A is thickly syndetic if any n-block, that is a block of n consecutive integers, appears in A syndetically: $\{i\colon\ [i, i+n) \cap \mathbb{N} \subset A\} \in \mathcal{F}_s$ for every $n > 0$. Denote by Γ_ℓ the set of all syndetic subsets with maximal gap at most ℓ, that is if $S \in \Gamma_\ell$, then $S + [-\ell, 0] \supset \mathbb{N}$.

For any $A \subset \mathbb{N}$, the *upper density* of A is defined by

$$\bar{d}(A) := \limsup_{n\to\infty} \frac{A \cap \{1, 2, \ldots, n\}}{n}. \tag{2.1}$$

Replacing lim sup with lim inf in (2.1), gives the definition of $\underline{d}(A)$, the *lower density* of A. We denote by $\mathcal{F}_{\underline{d}}$ (resp. $\mathcal{F}_{\bar{d}}$) the family of all subsets of $\mathbb{N}$ with positive lower (resp. upper) density. It is not hard to verify that if $B \in \mathcal{F}_{\underline{d}}^*$, then $\bar{d}(B) = 1$.

A family $\mathcal{F}$ has *Ramsey property* if for any $A \in \mathcal{F}$ and any finite partition $A = A_1 \cup \cdots \cup A_k$ there exists $1 \leqslant i \leqslant k$ such that $A_i \in \mathcal{F}$.

2.2. Topological dynamics

A (*topological*) *dynamical system* is a pair (X, f) consisting of a compact metric space (X, d) and a continuous surjection $f\colon X \to X$. If f is a homeomorphism, then we say that (X, f) is an *invertible dynamical system*. Fix any two nonempty open sets U, V and set $\mathcal{N}(U, V) := \{n\colon\ f^n(U) \cap V \neq \emptyset\}$. Similarly, if $x \in X$ then $\mathcal{N}(x, U) := \{n\colon\ f^n(x) \in U\}$. A dynamical system (X, f) is *transitive* if for any two nonempty open sets U, V we have $\mathcal{N}(U, V) \neq \emptyset$. If (X, f^n) is transitive for every $n = 1, 2, \ldots$ then we say that (X, f) is *totally transitive*. If for any nonempty open sets $\mathcal{N}(U, V) \in \mathcal{F}_t$ then (X, f) is *weakly mixing* and (X, f) is *mixing* if $\mathbb{N} \setminus \mathcal{N}(U, V)$ is finite. The above definitions of transitivity, weak mixing and mixing are one amongst many possible equivalent definitions [9].

A dynamical system (X, f) is *minimal* if the orbit of every point $x \in X$ is dense (see [9]). Equivalently, a dynamical system is minimal iff it does not contain any proper subsystem. Minimality concept is defined for a point as well: if $\mathcal{N}(x, U) \in \mathcal{F}_s$ for any open neighborhood U of x, then we say that x is a *minimal point*. It is clear that the closure of the orbit of a minimal point is a minimal dynamical system.

A pair (x, y) is *proximal* if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. A dynamical system is called *proximal system* if any pair of its distinct points is a proximal pair. It can be proved that a proximal system contains a unique minimal subsystem formed by a single fixed point p, and additionally, for any neighborhood U of p and any point $x \in X$ the set $\mathcal{N}(x, U)$ is thickly syndetic. A pair is *distal* if it is not proximal. A dynamical system is *equicontinuous* if for every $x \in X$ and every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \geqslant 0$.