J. Math. Anal. Appl. 417 (2014) 42–56

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The numerical range of a contraction with finite defect numbers

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ARTICLE INFO

Article history:
Received 9 May 2012
Available online 12 March 2014
Submitted by R. Curto

Keywords:
Contraction
Unitary dilation
Numerical range

ABSTRACT

An n-dilation of a contraction T acting on a Hilbert space $\mathcal{H}$ is a unitary dilation acting on $\mathcal{H} \oplus \mathbb{C}^n$. We show that if both defect numbers of T are equal to n, then the closure of the numerical range of T is the intersection of the closures of the numerical ranges of its n-dilations. We also obtain detailed information about the geometrical properties of the numerical range of T in case $n = 1$.

1. Numerical range and dilations

Assume that $\mathcal{H}$ is a complex separable Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We also use the notations $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. The spectrum of $T \in \mathcal{L}(\mathcal{H})$ is denoted by $\sigma(T)$, while the *numerical range* of T is defined by

$$W(T) := \bigl\{\langle Tx, x\rangle\colon\ x \in \mathcal{H},\ \|x\| = 1\bigr\}.$$

In this paper we are only concerned with contractions, that is, operators of norm at most 1. An arbitrary contraction can be decomposed as a direct sum of a unitary operator and a completely nonunitary contraction. The following basic properties of the numerical range of a contraction can be found, for instance, in [18, Ch. 1].

Proposition 1.1. *Let $T \in \mathcal{L}(\mathcal{H})$, $\|T\| \leqslant 1$. Then $W(T)$ is a convex subset of $\mathbb{C}$ which satisfies:*

(1) *$W(T) \subset \overline{\mathbb{D}}$ and $W(T) \subset \mathbb{D}$ if T is completely nonunitary.*

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[1] Supported in part by grants from the National Science Foundation (DMS-1065946).
[2] Supported in part by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0119.

http://dx.doi.org/10.1016/j.jmaa.2014.02.037
0022-247X/

(2) $\sigma(T) \subset \overline{W(T)}$;
(3) $\overline{W(T)} \cap \mathbb{T} = \sigma(T) \cap \mathbb{T}$.

We denote $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$; these are called the *defect operator* and the *defect space* of T respectively. The dimensions of $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are called the *defect indices* of T.

It is well-known that T admits unitary dilations; that is, there exist a space $\mathcal{K} \supset \mathcal{H}$ and a unitary operator $U \in \mathcal{L}(\mathcal{K})$ such that $T = PU|\mathcal{H}$, where P denotes the orthogonal projection in $\mathcal{K}$ onto $\mathcal{H}$. One can always take $\mathcal{K}$ to be $\mathcal{H} \oplus \mathcal{H}$; however, this is not the optimal choice when the defect spaces of T are of equal finite dimension n. Indeed, in this case there exist unitary dilations acting on $\mathcal{K} = \mathcal{H} \oplus \mathbb{C}^n$, and n is the smallest possible value of $\dim(\mathcal{K} \ominus \mathcal{H})$. We call such dilations *unitary n-dilations*.

It is obvious that $W(T) \subset W(U)$ for any unitary dilation U of T. Choi and Li [9] showed that, in fact,

$$\overline{W(T)} = \bigcap \big\{ \overline{W(U)}\colon\ U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } T \big\}, \tag{1.1}$$

thus answering a question raised by Halmos (see, for example, [19]). We note that when $\mathcal{H}$ is m-dimensional, the construction in [9] produces dilations which act on a space of dimension $2m$.

Assume now that $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = n$. We prove in Section 2 the stronger result

$$\overline{W(T)} = \bigcap \big\{ \overline{W(U)}\colon\ U \in \mathcal{L}(\mathcal{K}) \text{ is a unitary } n\text{-dilation of } T \big\}, \tag{1.2}$$

that is, we use only the most “economical” unitary dilations of T. This relation was proved earlier in special cases, namely for $\dim \mathcal{H} < \infty$ and $n = 1$ in [14], for $\dim \mathcal{H} < \infty$ and general n in [17], and for particular cases with $\dim \mathcal{H} = \infty$ in [6] and [4].

The remainder of the paper deals with the special case when $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = 1$. Partly as a consequence of the results in Section 2, we can investigate in detail the geometric properties of $W(T)$. The study in Section 3 does not use the functional model of a contraction, but this model is necessary for the results proved in later sections. Some properties are similar to those proved for finite dimensional contractions, but new phenomena appear for which we provide some illustrative examples.

Since the numerical range of a direct sum is the convex hull of the union of the numerical ranges of the summands, and the numerical range of a unitary operator is rather well understood, we always assume in the sequel that T is completely nonunitary. Also, if T has different defect indices, then $W(T) = \mathbb{D}$; so we restrict ourselves to studying completely nonunitary contractions with equal defect indices.

2. Unitary n-dilations

Let T be a completely nonunitary contraction with $\dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} = n < \infty$. The operator

$$\widetilde{T} = \begin{pmatrix} T & 0 \\ D_T & 0 \end{pmatrix} \tag{2.1}$$

is a partial isometry on $\mathcal{H} \oplus \mathcal{D}_T$ and $\sigma(\widetilde{T}) = \sigma(T) \cup \{0\}$. Both $\ker \widetilde{T}$ and $\ker \widetilde{T}^*$ have dimension n, and so any unitary operator $\Omega : \ker \widetilde{T} \to \ker \widetilde{T}^*$ determines a unitary n-dilation U_Ω of T by the formula

$$U_\Omega(x) = \begin{cases} \widetilde{T}x & \text{if } x \in \ker \widetilde{T}^\perp \\ \Omega x & \text{if } x \in \ker \widetilde{T}. \end{cases} \tag{2.2}$$

Conversely, any unitary n-dilation of T is unitarily equivalent to some U_Ω. The set $\sigma(U_\Omega) \setminus \sigma(T)$ consists of isolated Fredholm eigenvalues of U_Ω.