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Regularizing effects for the classical solutions to the Landau equation in the whole space

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ABSTRACT

The Landau equation describes the binary collisional effects (through long range coulombian interaction) in a plasma. In this paper, we prove that the known classical solutions to the Landau equation near Maxwellian in the whole space have a regularizing effect in all (time, space and velocity) variables, that is, become immediately smooth with respect to all variables.

1. Introduction and statement of the main result

We consider the following generalized Landau equation:

$$\partial_t F + v \cdot \nabla_x F = \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \psi(v-u)\big[F(u)\nabla_v F(v) - F(v)\nabla_u F(u)\big]\,du \right\}, \tag{1.1}$$

with the initial data $F(0,x,v) = F_0(x,v)$. Here $F(t,x,v) \geqslant 0$ is the distribution function for the particles at time $t \geqslant 0$, with spatial variable $x \in \mathbf{R}^3$ and velocity $v \in \mathbf{R}^3$. The non-negative matrix ψ is defined as

$$\psi^{ij}(v) = \left\{ \delta^{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^{\gamma+2}.$$

The index γ is a parameter leading to the standard classification of hard potential ($\gamma > 0$), the Maxwellian molecule ($\gamma = 0$) or soft potential ($\gamma < 0$), cf. [9,23]. The original Landau collision operator for the Coulombic interaction corresponds to the case $\gamma = -3$. In this paper, we restrict our discussion to the case $-3 \leqslant \gamma < -2$.

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It is well known that the classical Landau collision operator can be formally derived from the Boltzmann operator when the collision between particles become grazing. As in the Boltzmann equation, we denote a global Maxwellian by

$$M(v) = (2\pi)^{-3/2} e^{-|v|^2/2},$$

with the standard perturbation $F(t,x,v)$ to M as

$$F = M(v) + \sqrt{M} f.$$

Then the Landau equation (1.1) for $f(t,x,v)$ takes the form

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f), \qquad f(0,x,v) = f_0(x,v). \tag{1.2}$$

The Landau collision frequency is

$$\sigma^{ij} = \int_{\mathbf{R}^3} \psi^{ij}(v-u) M(u)\, du.$$

The linearized collision operator L in (1.2) is defined as [12,21]

$$Lf = -\frac{1}{\sqrt{M}} \{ Q(M, \sqrt{M} f) + Q(\sqrt{M} f, M) \} = -Af - Kf, \tag{1.3}$$

where

$$Af = \frac{1}{\sqrt{M}} Q(M, \sqrt{M} f) = \partial_i [\sigma^{ij} \partial_j f] - \sigma^{ij} \frac{v_i v_j}{4} f + \partial_i [\sigma^{ij} v_j] f, \tag{1.4}$$

$$\begin{aligned} Kf = \frac{1}{\sqrt{M}} Q(\sqrt{M} f, M) &= -\partial_i \left\{ M^{\frac12}(v) \left[\psi^{ij} * \left(M^{\frac12} \left(\partial_j f + \frac{v_j}{2} f \right) \right) \right] \right\} \\ &\quad + \frac{v_i}{2} M^{\frac12}(v) \left[\psi^{ij} * \left(M^{\frac12} \left(\partial_j f + \frac{v_j}{2} f \right) \right) \right], \end{aligned} \tag{1.5}$$

and the collision operator $\Gamma(f,g)$ is given by

$$\begin{aligned} \Gamma[f,g] &= M^{-1/2} Q[M^{1/2} f, M^{1/2} g] \\ &= \partial_i \{ [\psi^{ij} * (M^{1/2} f)] \partial_j g \} - \left\{ \left[\psi^{ij} * \left(\frac{v_i}{2} M^{1/2} f \right) \right] \right\} \partial_j g \\ &\quad - \partial_i \{ [\psi^{ij} * (M^{1/2} \partial_j f)] g \} + \left\{ \left[\psi^{ij} * \left(\frac{v_i}{2} M^{1/2} \partial_j f \right) \right] \right\} g, \end{aligned} \tag{1.6}$$

where we have used Einstein's summation for the repeated indices.

For notational simplicity, we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product in $\mathbf{R}_v^3$, with its L^2 norm given by $|\cdot|_2$, and $(\cdot,\cdot)$ is L^2 inner product in $\mathbf{R}_x^3 \times \mathbf{R}_v^3$ with corresponding L^2 norm $\|\cdot\|$. For $s \in \mathbf{R}$, $m \in \mathbb{Z}^+$, $p \geqslant 0$, we use the standard notation H^s or $W^{m,p}$ to denote the usual Sobolev space. Given any $k \in \mathbf{R}^3$ and function $f(x)$, let

$$\triangle_k f = f(x+k) - f(x). \tag{1.7}$$

We shall also use $(\cdot,\cdot)$ to denote the L^2 inner product in $\mathbf{R}_x^3 \times \mathbf{R}_v^3 \times \mathbf{R}_k^3$ with corresponding L^2 norm $|||\cdot|||$.