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Global existence of solutions to a parabolic–elliptic chemotaxis system with critical degenerate diffusion

Elissar Nasreddine

Institut de Mathématiques de Toulouse, Université de Toulouse, F-31062 Toulouse cedex 9, France

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ABSTRACT

This paper is devoted to the analysis of nonnegative solutions for a degenerate parabolic–elliptic Patlak–Keller–Segel system with critical nonlinear diffusion in a bounded domain with homogeneous Neumann boundary conditions. Our aim is to prove the existence of a global weak solution under a smallness condition on the mass of the initial data, thereby completing previous results on finite blow-up for large masses. Under some higher regularity condition on solutions, the uniqueness of solutions is proved by using a classical duality technique.

1. Introduction

Chemotaxis is the movement of biological organisms oriented towards the gradient of some substance, called the chemoattractant. The Patlak–Keller–Segel (PKS) model (see [14], [13] and [18]) has been introduced in order to explain chemotaxis cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density u, and a reaction diffusion equation for the chemoattractant concentration φ. It reads

$$(PKS)\quad \begin{cases} \partial_t u = \operatorname{div}(\nabla u^m - u \cdot \nabla\varphi) & x \in \Omega,\ t > 0, \\ -\Delta\varphi = u - \langle u \rangle & x \in \Omega,\ t > 0, \\ \langle \varphi(t) \rangle = 0 & t > 0, \\ \partial_\nu u = \partial_\nu \varphi = 0 & x \in \partial\Omega,\ t > 0, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases} \tag{1}$$

E-mail address: elissar_nd@hotmail.fr.

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where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, ν the outward unit normal vector to the boundary $\partial\Omega$ and $m \geqslant 1$. An important parameter in this model is the total mass M of cells, which is formally conserved through the evolution:

$$M = \langle u \rangle = \frac{1}{|\Omega|} \int_\Omega u(t,x)\,\mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega u_0(x)\,\mathrm{d}x. \tag{2}$$

Several studies have revealed that the dynamics of (1) depend sensitively on the parameters N, m and M. More precisely, if $N = 2$ and $m = 1$, it is well-known that the solutions of (1) may blow up in finite time if M is sufficiently large (see [18,16]) while solutions are global in time for M sufficiently small [18], see also the survey articles [4,11].

The situation is very different when $m = 1$ and $N \neq 2$. In fact, if $N = 1$, there is global existence of solutions of (1) whatever the value of the mass of initial data M, see [8] and the references therein. If $N \geqslant 3$, for all $M > 0$, there are initial data u_0 with mass M for which the corresponding solutions of (1) explode in finite time (see [16]). Thus, in dimension $N \geqslant 3$ and $m = 1$, the threshold phenomenon does not take place as in dimension 2, but we expect the same phenomenon when $N \geqslant 3$ and m is equal to the *critical* value $m = m_c = \frac{2(N-1)}{N}$. More precisely, we consider a more general version of (1) where the first equation of (1) is replaced by

$$\partial_t u = \operatorname{div}\bigl(\phi(u)\nabla u - u\nabla\varphi\bigr), \quad t > 0,\ x \in \Omega,$$

and the diffusivity ϕ is a positive function in $C^1([0,\infty[)$ which does not grow too fast at infinity. In [8], the authors proved that there is a critical exponent such that, if the diffusion has a faster growth than the one given by this exponent, solutions to (1) (with $\phi(u)$ instead of mu^{m-1}) exist globally and are uniformly bounded, see also [6,15] for $N = 2$. More precisely, the main results in [8] read as follows:

- If $\phi(u) \geqslant c(1+u)^p$ for all $u \geqslant 0$ and some $c > 0$ and $p > 1 - \frac{2}{N}$ then all the solutions of (1) are global and bounded.
- If $\phi(u) \leqslant c(1+u)^p$ for all $u \geqslant 0$ and some $c > 0$ and $p < 1 - \frac{2}{N}$ then there exist initial data u_0 such that

$$\lim_{t\to T} \bigl\|u(.,t)\bigr\|_\infty = \infty, \quad \text{for some finite } T > 0.$$

Except for $N = 2$, the critical case $m = \frac{2(N-1)}{N}$ is not covered by the analysis of [8]. Recently, Cieślak and Laurençot in [7] showed that if $\phi(u) \leqslant c(1+u)^{1-\frac{2}{N}}$ and $N \geqslant 3$, there are solutions of (1) blowing up in finite time when M exceeds an explicit threshold. In order to prove that, when $N \geqslant 3$ and $m = \frac{2(N-1)}{N}$, we have a threshold phenomenon similar to dimension $N = 2$ with $m = 1$, it remains to show that solutions of (1) are global when M is small enough. The goal of this paper is to show that this is indeed true, see Theorem 2.2 below.

By combining Theorem 2.2 with the blow-up result obtained in [7], we conclude that, for $N \geqslant 3$ and $m = \frac{2(N-1)}{N}$, there exists $0 < M_1 \leqslant M_2 < \infty$ such that the solutions of (1) are global if the mass M of the initial data u_0 is in $[0, M_1)$, and may explode in finite time if $M > M_2$. An important open question is whether $M_1 = M_2$ when Ω is a ball in $\mathbb{R}^N$ and u_0 is a radially symmetric function. Notice that, in the radial case, this result is true when $N = 2$ and $m = 1$, and the threshold value of the mass for blow-up is $M_1 = M_2 = 8\pi$, see [6,16,17,19]. Again, for $N = 2$ and $m = 1$, but for regular, connected and bounded domain, it has been shown that $M_1 = 4\pi = \frac{M_2}{2}$ (see [17,16] and the references therein). Such a result does not seem to be known for $N \geqslant 3$ and $m = \frac{2(N-1)}{N}$.

Still, in the whole space $\Omega = \mathbb{R}^N$ when the equation for φ in (1) is replaced by the Poisson equation $\varphi = E_N * u$, with E_N being the Poisson kernel, it has been shown in [9,5,2,21,22,3] that: