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Lower bounds on the eigenvalue gap for vibrating strings

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ABSTRACT

We provide lower bounds on the eigenvalue gap for vibrating strings with fixed endpoints depending only on qualitative properties of the density function. For example, if the density ρ is symmetric on the interval $[0,a]$, and if λ_1 and λ_2 are the first two eigenvalues of $u''(x)+\lambda\rho(x)u(x)=0$ in $(0,a)$ with $u(0)=u(a)=0$ boundary conditions, then

$$\lambda_2-\lambda_1>\max\left\{\frac{1}{\int_0^{a/2}(\frac{a}{2}-x)\rho(x)\,dx},\frac{\pi^2}{\rho_M a^2}\right\},$$

where $\rho_M=\max_{0\leqslant x\leqslant a}\rho(x)$. The ideas used also lead to applications in the case of monotone densities.

1. Introduction

The frequencies $\omega=\sqrt{\lambda}$ of a vibrating string with fixed endpoints are determined by the eigenvalues of the Sturm–Liouville problem

$$\begin{cases} u''(x)+\lambda\rho(x)u(x)=0 & \text{in } (0,a)\\ u(0)=u(a)=0 \end{cases} \tag{1.1}$$

where $\rho(x)$ is a positive continuous function and represents the mass density. As is well known, the eigenvalues of (1.1) form a strictly increasing sequence of positive numbers which depend on the density $\rho(x)$. We denote them accordingly by

$$0<\lambda_1[\rho]<\lambda_2[\rho]<\lambda_3[\rho]<\cdots.$$

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Bounds on ratios of eigenvalues, especially the ratio of the first two eigenvalues, have long been of interest (see e.g., [2,3,5,6] and references therein). Of related interest is the problem of obtaining optimal bounds on gaps of eigenvalues. Huang [7] has considered the gap of the first two eigenvalues of (1.1) for certain classes of symmetric densities and established several comparison results in the case of single-well densities and in the case of double-well densities. His approach is based on a variational principle for the gap.

The purpose of this paper is to provide lower bounds on the eigenvalue gap for (1.1). Here we will present a very elementary device which is also quite useful. The function $\varphi(x) = u_2(x)/u_1(x)$, where $u_1(x)$ and $u_2(x)$ are the first two eigenfunctions of (1.1), will play a key role in our discussion. Our lower bounds depend only on qualitative properties of the density function. For example, if the density ρ is symmetric on the interval $[0, a]$, then

$$\lambda_2[\rho] - \lambda_1[\rho] > \max\left\{ \frac{1}{\int_0^{a/2} (\frac{a}{2} - x)\rho(x)\,dx}, \frac{\pi^2}{\rho_M a^2} \right\},$$

where $\rho_M = \max_{0 \leqslant x \leqslant a} \rho(x)$. We also apply our lower bound techniques to strings with monotone densities.

An example is included in the last section, where we consider the eigenvalue problem of vibrating strings in the interval $(-a, a)$ with density $\rho(x) = |x|^m$, $m \geqslant 0$. We shall solve this problem by means of Bessel functions. We shall also compare our lower bound results with the actual eigenvalue gap.

2. Preliminaries

Let $u_n(x)$ be the nth eigenfunction of (1.1) corresponding to $\lambda_n[\rho]$, normalized so that

$$\int_0^a \rho(x) u_n^2(x)\,dx = 1.$$

It is known that $u_n(x)$ has exactly $(n-1)$ zeros in the open interval $(0, a)$. We may assume that $u_1(x) > 0$ on $(0, a)$, and that $u_2(x) > 0$ on $(0, x_0)$ and $u_2(x) < 0$ on (x_0, a), where $u_2(x_0) = 0$. Since $u_1''(x) = -\lambda_1[\rho]\rho(x)u_1(x) < 0$ on $(0, a)$, $u_1(x)$ is strictly concave on $(0, a)$. It follows that $u_1(x)$ has a unique critical point in $(0, a)$ so that for some $c_0 \in (0, a)$,

$$u_1'(x) > 0 \quad \text{on } (0, c_0) \quad \text{and} \quad u_1'(x) < 0 \quad \text{on } (c_0, a). \tag{2.1}$$

Let

$$\Gamma[\rho] = \lambda_2[\rho] - \lambda_1[\rho]$$

be the gap of the first two eigenvalues of (1.1), and define

$$\varphi(x) = \frac{u_2(x)}{u_1(x)}.$$

It was shown in [6] that $\varphi(x)$ is strictly decreasing on $(0, a)$. Also, by the differential equation and l'Hôpital's rule, we have

$$\varphi'(0) = \varphi'(a) = 0 \tag{2.2}$$

$$\varphi''(x) = -\Gamma[\rho]\rho(x)\varphi(x) - 2\varphi'(x)(\log u_1)'(x). \tag{2.3}$$