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Functional inequalities for generalized inverse trigonometric and hyperbolic functions

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ABSTRACT

Various miscellaneous functional inequalities are deduced for the so-called generalized inverse trigonometric and hyperbolic functions. For instance, functional inequalities for sums, difference and quotient of generalized inverse trigonometric and hyperbolic functions are given, as well as some Grünbaum inequalities with the aid of the classical Bernoulli inequality. Moreover, by means of certain already derived bounds, bilateral bounding inequalities are obtained for the generalized hypergeometric ${}_3F_2$ Clausen function.

1. Introduction and main results

For given complex numbers a, b and c with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* ${}_2F_1$ is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n \geqslant 0} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here (a, n) is the Pochhammer symbol (rising factorial) $(\cdot, n) : \mathbb{C} \to \mathbb{C}$, defined by

$$(z, n) = \frac{\Gamma(z + n)}{\Gamma(z)} = \prod_{j=1}^{n} (z + j - 1)$$

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for $n \in \mathbb{Z}$, see [1]. Special functions, such the classical *gamma function* Γ, the *digamma function* ψ and the *beta function* $B(\cdot,\cdot)$ have close relation with hypergeometric function. These functions for $x, y > 0$ are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively.

The eigenfunction $\sin_p$ of the of the so-called one-dimensional p-Laplacian problem [11]

$$-\Delta_p u = -\left(\left|u'\right|^{p-2} u'\right)' = \lambda |u|^{p-2} u, \qquad u(0) = u(1) = 0, \quad p > 1,$$

is the inverse function of $F : (0,1) \to (0, \frac{\pi_p}{2})$, defined as

$$F(x) = \operatorname{arcsin}_p(x) = \int_0^x \left(1 - t^p\right)^{-\frac{1}{p}}\, dt,$$

where

$$\pi_p = \frac{2}{p} \int_0^1 (1-s)^{-\frac{1}{p}} s^{\frac{1}{p}-1}\, ds = \frac{2}{p} B\left(1 - \frac{1}{p}, \frac{1}{p}\right) = \frac{2\pi}{p \sin(\frac{\pi}{p})}.$$

The function arcsin_p is called the generalized inverse sine function, and coincides with usual inverse sine function for $p = 2$. Similarly, the other generalized inverse trigonometric and hyperbolic functions $\operatorname{arccos}_p : (0,1) \to (0, \frac{\pi}{2})$, $\operatorname{arctan}_p : (0,1) \to (0, b_p)$, $\operatorname{arcsinh}_p : (0,1) \to (0, c_p)$, $\operatorname{arctanh}_p : (0,1) \to (0, \infty)$, where

$$b_p = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right) = 2^{-\frac{1}{p}} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right), \qquad c_p = \left(\frac{1}{2}\right)^{\frac{1}{p}} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}, \frac{1}{2}\right),$$

are defined as follows

$$\operatorname{arccos}_p(x) = \int_0^{(1-x^p)^{\frac{1}{p}}} \left(1 - t^p\right)^{-\frac{1}{p}}\, dt,$$

$$\operatorname{arctan}_p(x) = \int_0^x \left(1 + t^p\right)^{-1}\, dt,$$

$$\operatorname{arcsinh}_p(x) = \int_0^x \left(1 + t^p\right)^{-\frac{1}{p}}\, dt,$$

$$\operatorname{arctanh}_p(x) = \int_0^x \left(1 - t^p\right)^{-1}\, dt.$$

These functions are the generalizations of the usual elementary inverse trigonometric and hyperbolic functions and are the inverse of the so-called generalized trigonometric and hyperbolic functions, introduced by P. Lindqvist [16], see also [10,12,17,23] for more details. Recently, there has been a vivid interest on the