J. Math. Anal. Appl. 417 (2014) 361–375

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A two-point boundary value problem arising in boundary layer theory ☆

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ARTICLE INFO

Article history:
Received 17 May 2013
Available online 24 March 2014
Submitted by Goong Chen

Keywords:
Boundary layer flow
Two-point boundary value problem
Singular initial value problem
Normal solution
Positive solution

ABSTRACT

A two-point boundary value problem of second order is restudied, which describes both a similarity boundary-layer flow induced by a moving permeable plane surface and a self-similar free convection boundary-layer flow over a vertical permeable flat plate embedded in a fluid-saturated porous medium. It is proved that there exists a $\lambda_{\min} \in [1, 2/\sqrt{3}]$ such that this problem has a unique normal solution $\phi(\eta;\lambda)$ for all $\lambda \geqslant \lambda_{\min}$ and the solution $\phi(\eta;\lambda)$ is strictly decreasing with respect to both $\eta \geqslant 0$ and $\lambda \geqslant \lambda_{\min}$; moreover, $\phi'(0;\lambda_{\min}) = 0$ and $\phi'(0;\lambda)$, as a function of λ, is strictly decreasing for $\lambda \geqslant \lambda_{\min}$.

1. Introduction

In this paper, we present a rigorous mathematical analysis for the two-point boundary value problem involving a positive parameter λ,

$$\begin{cases} \phi''(\eta) + \lambda\phi'(\eta) = -\phi^2(\eta), & \eta \geqslant 0, \\ \phi(0) = 1, \qquad \phi(\infty) = 0, \end{cases} \tag{1}$$

which has already been examined in Refs. [5,6]. Here and henceforth, the primes denote differentiation with respect to the independent variable η (or t or x).

In boundary layer theory, BVP (1) describes both a similarity boundary-layer flow induced by a moving permeable plane surface in a quiescent incompressible Newtonian fluid (see [5]) and a self-similar free convection boundary-layer flow over a heated vertical permeable flat plate with lateral mass fmedium (see [6]). Obviously, the above second-order nonlinear autonomous differential equation also describes the one-dimensional motion of a point particle of coordinate ϕ, mass $M = 1$ and potential energy $E = \phi^3/3$, in the presence of a viscous friction with viscous friction coefficient λ. The dimensionless variable η plays the

☆ The project was supported by the Natural Science Foundation of Fujian Province of China (No. 2011J01019).
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http://dx.doi.org/10.1016/j.jmaa.2014.03.051
0022-247X/

role of time parameter. With the aid of this analogy, accompanied by numerical calculations, Magyari, Pop and Keller deduced in both [5] and [6] the conclusion that there exists a $\lambda_{\min} = 1.079131$ such that BVP (1) has a unique positive solution for $\lambda = \lambda_{\min}$ and multiple positive solutions for all $\lambda > \lambda_{\min}$. From the conclusion, we know that BVP (1) is not a well-posed mathematical model. Therefore, this conclusion can not help us to understand correctly the similarity boundary-layer flows described by BVP (1). To look for a solution that possesses physical meaning, BVP (1) should be replenished with some constraint conditions. In order to study BVP (1) and to obtain a new conclusion, we need to introduce the following important concept.

Definition 1. A function $\phi(\eta) \in C^2[0,\infty)$ is called a normal solution to BVP (1) if it is a positive solution to BVP (1) that satisfies the following two constraint conditions

$$\phi'(0) \leqslant 0 \quad \text{and} \quad \int_0^\infty \phi(\eta)\,d\eta < +\infty.$$

As far as the physical problem considered in [5] is concerned, η is the similarity variable, $\phi(\eta)$ is the dimensionless downstream velocity, the positive parameter λ plays the role of a dimensionless suction velocity of the ambient fluid, and $\phi'(0)$ is the dimensionless skin friction acting on the moving plane surface, and the value of $\int_0^\infty \phi(\eta)\,d\eta$ represents the dimensionless displacement thickness of the boundary layer. Judging from these, the first constraint condition prescribes that the moving plane surface should experience a drag motion when $\phi'(0) < 0$ or a dragless motion when $\phi'(0) = 0$, and the second constraint condition prescribes that the dimensionless displacement thickness of boundary layer must be bounded. From the point of view of fluid mechanics, the two constraint conditions are fair and reasonable; moreover, they guarantee that a normal solution is certainly a solution possessing physical meaning. Concerning the normal solution, we will prove the following theorem.

Theorem 1. *There exists a $\lambda_{\min} \in [1, \frac{2}{\sqrt{3}}]$ such that for all $\lambda \geqslant \lambda_{\min}$, BVP (1) admits one and only one normal solution $\phi(\eta;\lambda)$. The normal solution $\phi(\eta;\lambda)$ has the following five properties:*

(i) *$\phi(\eta;\lambda) \in C^\infty[0,+\infty)$;*
(ii) *$\phi(\eta;\lambda) > e^{-\lambda\eta}$ and $\phi'(\eta;\lambda) < 0$ for all $\eta > 0$ and $\lambda \geqslant \lambda_{\min}$;*
(iii) *for each fixed $\eta > 0$, the normal solution $\phi(\eta;\lambda)$, as a function of λ, is continuous and strictly decreasing on $[\lambda_{\min},\infty)$;*
(iv) *$\phi'(0;\lambda)$, as a function of λ, is continuous and strictly decreasing on $[\lambda_{\min},\infty)$ and $\phi'(0;\lambda_{\min}) = 0$;*
(v) *$\phi'(\infty;\lambda) = 0$ and $\lim_{\eta\to\infty} \frac{\phi'(\eta;\lambda)}{\phi(\eta;\lambda)} = \lim_{\eta\to\infty} \frac{\phi''(\eta;\lambda)}{\phi'(\eta;\lambda)} = -\lambda$.*

Theorem 1 declares that the two-point boundary value problem with two constraint conditions $\phi'(0) \leqslant 0$ and $\int_0^\infty \phi(\eta)\,d\eta < +\infty$ is really a well-posed mathematical model. Of course, it can help us to understand correctly the similarity boundary-layer flow described by BVP (1).

In order to prove Theorem 1, we need to solve the singular initial value problem

$$\begin{cases} w'(t) = \lambda - \dfrac{t^2}{w(t)}, & t > 0, \\ w(0) = 0, \qquad w'(0) = \lambda > 0. \end{cases} \tag{2}$$

Owing to $w(0) = 0$, the initial point $t = 0$ is a singular point and hence the "extra" initial condition $w'(0) = \lambda > 0$ is absolutely necessary. In Section 3, we will prove that the singular initial value problem (2) has a unique positive solution defined on $[0, \lambda^2 R)$, where $R \in [\frac{3}{4}, 1]$.