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First stability eigenvalue characterization of CMC Hopf tori into Riemannian Killing submersions

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A B S T R A C T

We find out upper bounds for the first eigenvalue of the stability operator for compact constant mean curvature orientable surfaces immersed in a Riemannian Killing submersion. As a consequence, the strong stability of such surfaces is studied. We also characterize constant mean curvature Hopf tori as the only ones attaining the bound in certain cases.

1. Introduction

Let $\psi : \Sigma^2 \to M^3$ be an isometric immersion of a compact orientable surface in a three dimensional oriented Riemannian manifold. Fixed a unit normal vector field N globally defined on Σ, we will denote by A the second fundamental form with respect to N of the immersion and by H its mean curvature.

Every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ induces a normal variation ψ_t of the immersion ψ, with variational normal field fN and first variation of the area functional $\mathcal{A}(t)$ given by $\delta_f \mathcal{A} = \mathcal{A}'(0) = -2\int_\Sigma fH$. As a consequence, minimal surfaces ($H = 0$) are characterized as the critical points of the area functional whereas constant mean curvature (CMC) surfaces are the critical points of the area functional restricted to smooth functions f such that $\int_\Sigma f = 0$, which means that the variation leaves constant the volume enclosed by the surface.

For such critical points, the stability of the corresponding variational problem is given by the second variation of the area functional,

$$\delta_f^2 \mathcal{A} = \mathcal{A}''(0) = -\int_\Sigma fJf,$$

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with $Jf = \Delta f + (|A|^2 + \overline{\mathrm{Ric}}(N,N))f$, where Δ stands for the Laplacian operator on Σ and $\overline{\mathrm{Ric}}$ denotes the Ricci curvature of M. The surface Σ is said to be *strongly stable* if $\delta_f^2 \mathcal{A} \geqslant 0$, for every $f \in \mathcal{C}^\infty(\Sigma)$. The operator $J = \Delta + |A|^2 + \overline{\mathrm{Ric}}(N,N)$, which is a Schrödinger operator, is known as the *Jacobi* or *stability operator* of the surface. The spectrum of J

$$\mathrm{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \cdots\}$$

consists of an unbounded increasing sequence of eigenvalues λ_k with finite multiplicities. Moreover, the first eigenvalue is simple (multiplicity one) and it satisfies the following min–max characterization

$$\lambda_1 = \min\left\{\frac{-\int_\Sigma fJf}{\int_\Sigma f^2}\colon\ f \in \mathcal{C}^\infty(\Sigma),\ f \neq 0\right\}. \tag{1}$$

In terms of the spectrum, Σ is strongly stable if and only if $\lambda_1 \geqslant 0$.

Observe that with our criterion, a real number λ is an eigenvalue of J if and only if $Jf + \lambda f = 0$ for some smooth function $f \in \mathcal{C}^\infty(\Sigma)$, $f \neq 0$.

In 1968, Simons [13] found out an estimate for the first eigenvalue of J on any compact minimal hypersurface in the standard sphere. In particular, for minimal surfaces in the 3-sphere he proved that $\lambda_1 = -2$ if the surface is totally geodesic and $\lambda_1 \leqslant -4$ otherwise. Later on, Wu [15] characterized the equality by showing that it holds only for the minimal Clifford torus. In the last decade, Perdomo [10] gave a new proof of this spectral characterization by getting an interesting formula that relates the first eigenvalue λ_1, the genus of the surface, the area and a simple invariant. Alías, Barros and Brasil [1] extended Wu and Perdomo's results to the case of CMC hypersurfaces in the standard sphere, characterizing some CMC Clifford tori. Very recently, Chen and Wang [5] have just given optimal estimates for λ_1 for linear Weingarten hypersurfaces in the sphere characterizing the equality.

In [2] the authors study the same problem in homogeneous Riemannian 3-manifolds, in particular in Berger spheres, finding out upper bounds for λ_1 for compact CMC surfaces immersed into such manifolds. They also get a characterization of CMC Hopf tori in certain Berger spheres and in the product $\mathbb{S}^2 \times \mathbb{S}^1$.

Homogeneous Riemannian 3-manifolds are a special kind of manifolds belonging to a more general structure with many more interesting examples: Riemannian Killing submersions. They are Riemannian 3-manifolds which fiber over a Riemannian surface and whose fibers are the trajectories of a unit Killing vector field. The study of immersed surfaces into such manifolds is a subject of increasing interest (see [6–8,12] and references therein). In these manifolds, there also exist Hopf tori, which are obtained as the total lift of closed curves by means of the submersion (if it has compact fibers). These surfaces are flat and have constant mean curvature when the curve has constant curvature. They first appear in [11] for the Hopf fibration, where they play a key role, and they have been studied in several works (e.g. [3] and [4]).

In this paper, we extend the results in [2] to Riemannian Killing submersions giving some estimates for λ_1 for CMC compact orientable surfaces immersed in such manifolds. As a consequence, the strong stability of such surfaces is studied. We also characterize CMC Hopf tori as the only ones attaining the upper bound in certain cases.

2. Riemannian Killing submersions

Let M^3 be a three dimensional oriented Riemannian manifold and $\pi : M \to B$ a Killing submersion over a surface B^2, i.e. a Riemannian submersion whose vertical unit vector field ξ is a unit Killing vector field on M (hence the fibers are geodesics). In this situation M is called a Riemannian Killing submersion. We remind that a vector field on M is vertical if it is always tangent to fibers and horizontal if it is always orthogonal to fibers. If $\overline{\nabla}$ stands for the Levi-Civita connection of M, we have