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Asymptotic behavior of positive solution in an annulus

Zuji Guo

School of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

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ABSTRACT

This paper concerns the positive solutions of superlinear elliptic equation with Hardy potential:

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2} u = |u|^{p-2} u \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is an annulus of $\mathbb{R}^N$, $N \geqslant 2$. We obtain qualitative properties of solutions as $p \to +\infty$.

1. Introduction and main results

In this paper we consider the following problem:

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2} u = |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$. Eq. (1.1) with $\mu = 0$ is the famous Lane–Emden–Fowler equation. In this case, solutions of this equation have been studied extensively in the last century. Define

$$2^* = \begin{cases} \infty & \text{if } N = 2, \\ \frac{2N}{N-2} & \text{if } N \geqslant 3. \end{cases}$$

If $2 < p < 2^*$ it is well known that (1.1) has infinitely many solutions by elegant symmetric mountain pass lemma [1]. If $p = 2^*$ and Ω is a star-shaped domain then $u = 0$ is the only solution of (1.1), as seen from Pohozaev's identity [24]. But if Ω has a nontrivial topology or if the equation is suitably perturbed, the results of [2] and [4] hold for positive solutions. For $p > 2^*$, (1.1) is more involved since in this case no Sobolev imbedding is available and variational methods cannot be directly applied to (1.1). While in various

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cases solutions of (1.1) with supercritical exponent have been constructed (see for example [7,8,23]) and uniqueness of positive and sign-changing radial solutions has been proved (see for example [6,9,16,17,22]), the question concerning properties of solutions of (1.1) remains largely open.

In the special case when Ω is an annulus it is easy to prove that a radial positive solution always exists as long as $p > 2$ (see [18]), and this solution is unique [22]. Moreover, exploiting the invariance of the annulus with respect to different symmetry groups, several authors were able to prove the existence of nonradial positive solutions in annuli $A_R = \{x \in \mathbb{R}^N\colon R < |x| < R+1\}$ for R sufficiently large (see [5,21,20]).

Observing the fact that the hypothesis on μ deals with the existence of the solution and not with the compactness of the embedding: the Sobolev space $H^1_{0,rad} = \{u \in H^1_0(\Omega) \mid u(x) = u(|x|)\}$ is compactly embedded in $L^p(\Omega)$ for every $p > 2$ when Ω is an annulus and $\mu > \frac{-1}{4}[(\frac{2\pi}{\ln b/a})^2 + (N-2)^2]$.

In this paper, inspired by the paper [13], we study the asymptotic behavior of those positive solutions for (1.1) as $p \to +\infty$. When $\mu \neq 0$, the computation is more involved than $\mu = 0$ as in [13], in particular, when $\mu < -\frac{(N-2)^2}{4}$. We will see the asymptotic behavior is essential different as the parameter μ changes, therefore we generalize the results of [13].

We also observe that the radial nondegeneracy of the positive solutions is proved in [3, Proposition 2.1], and the degeneracy points for $\mu = 0$ are characterized in [12, Lemma 2.3]. Here we will give a different proof of the radial nondegeneracy of the positive solutions for (1.1).

These nondegenerate solutions were used in [3,6,9] as building blocks to construct solutions of elliptic equations with supercritical exponent on a bounded domain with a very small hole or expanding annuli.

We hope this interesting analysis gives some useful ideas to deduce existence results to (1.1) in "annular-type" domains.

In the following of the paper Ω will denote the annulus $\Omega = \{x \in \mathbb{R}^N\colon 0 < a < |x| < b\}$ and ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

We define

$$\lambda_1 = \frac{1}{4}\left[\left(\frac{2\pi}{\ln b/a}\right)^2 + (N-2)^2\right].$$

Our first results concerns the convergence of the solution u_p of (1.1).

Theorem 1.1. *Let u_p be the unique positive radial solution of (1.1).*

(i) *If $\mu \geqslant -\frac{(N-2)^2}{4}$, then as $p \to +\infty$,*

$$u_p(|x|) \to U(|x|) \quad \text{in } C^0(\overline{\Omega}). \tag{1.2}$$

The function U is defined by

$$U(|x|) = \frac{2}{a^{-\sqrt{\Delta}} - b^{-\sqrt{\Delta}}}|x|^{-\lambda}\begin{cases} a^{-\sqrt{\Delta}} - |x|^{-\sqrt{\Delta}}, & a \leqslant |x| \leqslant r_0, \\ |x|^{-\sqrt{\Delta}} - b^{-\sqrt{\Delta}}, & r_0 \leqslant |x| \leqslant b, \end{cases} \tag{1.3}$$

where

$$r_0 = \left[\frac{a^{-\sqrt{\Delta}} + b^{-\sqrt{\Delta}}}{2}\right]^{-\frac{1}{\sqrt{\Delta}}}, \tag{1.4}$$

and

$$\lambda = \frac{N-2-\sqrt{\Delta}}{2}, \quad \Delta = (N-2)^2 + 4\mu. \tag{1.5}$$

(1.3) and (1.4) are understood in the limit's sense when $\Delta = 0$.