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Convergence of nonlocal diffusion models on lattices

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ABSTRACT

In this paper we look at models of nonlocal (or anomalous) diffusion which are defined on subsets of the lattice $\epsilon\mathbb{Z}^n$, for some $\epsilon > 0$, and ask if they can be approximated by continuum models. The answer is given by an operator semigroup convergence theorem. As an application, we establish hypotheses under which a discrete model of nonlocal diffusion satisfying an absorbing boundary condition has a continuum limit which is conservative.

1. Introduction

The study of nonlocal diffusion (also called anomalous diffusion) has recently emerged as an important area of scientific research, with applications in such disparate areas as groundwater hydrology (see Meerschaert and Sikorskii [11]), optimal search theory (Raposo et al. [12]), and financial market modeling (Mantegna [10]). Roughly speaking, nonlocal diffusion occurs when a "particle" moves in a way similar to a simple random walk but has different asymptotic properties because it occasionally takes very large jumps. It is well known that, under appropriate hypotheses, simple random walks can be approximated by continuum models governed by the heat equation (see Burdzy and Chen [3], Lin and Segel [9]). The aim of this paper is to prove some related results for models of nonlocal diffusion.

As a starting point, consider the system of equations

$$\frac{d}{dt}p(x,t) = \sum_{y\in\mathbb{Z}^n\setminus\{x\}} \frac{\mathcal{C}(p(y,t)-p(x,t))}{|y-x|^{n+\alpha}}, \quad x\in\mathbb{Z}^n \tag{1}$$

where $\mathcal{C} > 0$ and $\alpha \in (0,2)$. The solution to this system gives the probability $p(x,t)$ that a randomly moving particle is at the point x at time t, given an appropriate initial condition and given that the position X_t of

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the particle at time t is a continuous time Markov chain with transition probabilities given by

$$\mathrm{P}(X_{t+s} = y | X_t = x) = \frac{s\mathcal{C}}{|y-x|^{n+\alpha}} + o(s).$$

For a given $\epsilon > 0$, if we rescale space and time so that the movement of the particle is described by $X^{(\epsilon)}$, where $X_t^{(\epsilon)} = \epsilon X_{\epsilon^{-\alpha}t}$, then one can easily verify that the probability $p^\epsilon(x,t)$ of finding the particle at x at time t satisfies

$$\frac{d}{dt}p^\epsilon(x,t) = \sum_{y\in\epsilon\mathbb{Z}^n\setminus\{x\}} \frac{\mathcal{C}(p^\epsilon(y,t) - p^\epsilon(x,t))\epsilon^n}{|y-x|^{n+\alpha}}, \quad x \in \epsilon\mathbb{Z}^n. \tag{2}$$

The results of Husseini and Kassmann [7] show that as $\epsilon \to 0$, $X^{(\epsilon)}$ converges to a stochastic process governed by a fractional diffusion equation.

A natural generalization of (2) for subsets E of $\epsilon\mathbb{Z}^n$ is the system

$$\frac{d}{dt}p(x,t) = \sum_{y\in E\setminus\{x\}} \frac{\mathcal{C}(p(y,t) - p(x,t))\epsilon^n}{|y-x|^{n+\alpha}}, \quad x \in E. \tag{3}$$

This model has appeared in certain applied contexts (as a special case of the model of human mobility in Brockmann [2] and as a model of anomalous diffusion in Condat, Rangel and Lamberti [4]). In what follows we will study this model, as well as an altered version involving an absorbing boundary condition. We will give hypotheses under which they converge, in a sense to be made precise in the next section, as $\epsilon \to 0$. This is a continuation of previous work with Seidman in [15].

2. Formal construction of the models and statement of the main results

To motivate all the definitions below, let us briefly summarize the elements of the argument to follow. Given a lattice $\epsilon\mathbb{Z}^n$ and a bounded open set $U \subset \mathbb{R}^n$, we consider a family of n-dimensional cubes $S_1, \dots, S_m$ which cover U. We choose the cubes so that each one is centered at a lattice point in $\epsilon\mathbb{Z}^n$, has non-empty intersection with U, and has volume ϵ^n. Letting z_i denote the lattice point at the center of S_i, δ_{z_i} denote the Dirac measure centered at z_i, and $\mathbb{1}_{S_i}$ denote the characteristic function for S_i, we see that each probability measure $\mu = \sum_{i=1}^m c_i\delta_{z_i}$ on $\{z_1, \dots, z_m\}$ can be associated with a probability density $v = \frac{1}{\epsilon^n}\sum_{i=1}^m c_i\mathbb{1}_{S_i}$ on $\bigcup_{i=1}^m S_i$, which satisfies $\mu(\{z_i\}) = \int_{S_i} v(x)\,dx$ for all i. Using this correspondence, we can identify the transition semigroup of a given Markov chain on $\{z_1, \dots, z_m\}$ with a semigroup acting on a space of piecewise constant functions $g : \bigcup_{i=1}^m S_i \to \mathbb{R}$. For sufficiently small ϵ, the latter semigroup will approximate some limiting semigroup acting on $L^2(U)$, and this is the continuum-limit model.

We can now proceed with the detailed construction of the models. In everything that follows, $\alpha \in (0,2)$, $\mathcal{C}$ is a positive constant and $(\epsilon_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers such that $\epsilon_k \downarrow 0$. We will assume U is a bounded open subset of $\mathbb{R}^n$ satisfying the *segment property*: for each x contained in the boundary ∂U of U, there is a neighborhood N_x of x in $\mathbb{R}^n$ and a vector y_x, distinct from the zero vector $\mathbf{0}$, such that $z + ty_x \in U$ for every $z \in \overline{U} \cap N_x$ and $t \in (0,1)$. (All bounded Lipschitz open sets satisfy the segment property, see Grisvard [6, Theorem 1.2.2.2].) We assume in addition that

$$\lim_{\xi\downarrow 0} \lambda\bigl(\bigl\{x \in \mathbb{R}^n\colon d(x, \partial U) < \xi\bigr\}\bigr) = 0$$

where λ denotes the Lebesgue measure on $\mathbb{R}^n$. For each k, fix a bijection $\mathbb{N} \to \epsilon_k\mathbb{Z}^n : i \mapsto z_{ki}$. Define the cube