J. Math. Anal. Appl. 415 (2014) 14–24

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global attractors for the complex Ginzburg–Landau equation

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A R T I C L E   I N F O

Article history:
Received 8 May 2013
Available online 24 January 2014
Submitted by J. Morgan

Keywords:
Global attractor
Complex Ginzburg–Landau equations
Sobolev compactness embedding theorem
Interpolation inequality

A B S T R A C T

This paper concerns the long-time behavior of the following complex Ginzburg–Landau equations

$$\frac{\partial u}{\partial t}-(\lambda+i\alpha)\Delta u+(\kappa+i\beta)|u|^{p-2}u-\gamma u=0$$

without any restriction on $p>2$ under the assumptions (1.4). We first prove the well-posedness of strong solutions for the complex Ginzburg–Landau equations, and then the existence of absorbing sets in $L^2(\Omega)$, $H_0^1(\Omega)\cap L^p(\Omega)$ and $H^2(\Omega)\cap L^{2(p-1)}(\Omega)$, respectively, for the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by (1.1)–(1.3) is established. Finally, we prove the existence of global attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ for the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by (1.1)–(1.3) by the Sobolev compactness embedding theorem and prove the existence of global attractor in $L^p(\Omega)$ for the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by (1.1)–(1.3) using interpolation inequality.

1. Introduction

In this paper, we will consider the existence of global attractors in $L^2(\Omega)$, $L^p(\Omega)$ and $H_0^1(\Omega)$ of the following initial–boundary value problem for the complex Ginzburg–Landau equation

$$\frac{\partial u}{\partial t}-(\lambda+i\alpha)\Delta u+(\kappa+i\beta)|u|^{p-2}u-\gamma u=0,\quad (x,t)\in\Omega\times\mathbb{R}^+,\tag{1.1}$$

$$u=0,\quad (x,t)\in\partial\Omega\times\mathbb{R}^+,\tag{1.2}$$

$$u(x,0)=u_0(x),\quad x\in\Omega,\tag{1.3}$$

where $\Omega\in\mathbb{R}^n$ $(n\geqslant 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $t\geqslant 0$, $i=\sqrt{-1}$, $\lambda>0$, $\kappa>0$, α, $\beta\in\mathbb{R}$, $\gamma>0$, the exponent $p>2$ are constants and u is a complex-valued unknown function.

The complex Ginzburg–Landau equation is known as an important model equation describing spatial pattern formation or the amplitude evolution of instability in non-equilibrium fluid dynamical systems

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.059

(from the development of Tollmien–Schlichting waves in plane Poiseuille flows, to the appearance of Taylor vortices in the flow between counterrotating circular cylinders, and the study of chemical systems) as well in the theory of phase transitions and superconductivity (see [4,13,14]). In its special cases, the equation meets the nonlinear Schrödinger equation which is recently studied as various type equations with generalized nonlinear term. Therefore more and more mathematics have paid attention to the complex Ginzburg–Landau type equation in both theoretical physics and mathematics.

In the last several decades, many authors have studied extensively the initial–boundary value problem for the complex Ginzburg–Landau equation by different methods (see [2,3,6,8–10,17,16,18,15,21,22]). For example, Ginibre and Velo [8] proved the global existence of weak solutions to (1.1) for $u_0 \in L^2(\Omega)$ by a Faedo–Galerkin method with $p > 2$. Okazawa and Yokota [18] gave a monotonicity method which is the simplest way to prove the existence and uniqueness simultaneously and proved the global existence of unique strong solutions to (1.1) for $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2p}(\Omega)$ by compactness methods without any upper restriction on $p \geqslant 2$ but with the restriction $\frac{|\beta|}{\kappa} \leqslant \frac{p-2}{2\sqrt{p-1}}$, which can also even guarantee the uniqueness of mild solutions. Yokota [22] proved the global existence of unique strong solutions for the complex Ginzburg–Landau type equation with the nonlinearity $\kappa|u|^{p-2}u + i\beta|u|^{r-2}u$ $(q > r \geqslant 2)$ for $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ under the restriction imposed on $\frac{|\alpha|}{\lambda} \leqslant \frac{p-2}{2\sqrt{p-1}}$. Under the assumption $(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}) \in CGL(\frac{1}{c_p})$, where

$$CGL(y_0) := \left\{(x,y) \in \mathbb{R}^2\colon\ xy \geqslant 0 \text{ or } \frac{|xy|-1}{|x|+|y|} < y_0\right\},$$

$$c_p = \frac{p-2}{2\sqrt{p-1}},$$

Clément et al. [3] proved the global existence of strong solutions $u \in C([0,\infty); L^2(\mathbb{R}^n)) \cap C([0,\infty); H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$ for the complex Ginzburg–Landau equation in $\mathbb{R}^n$ with initial date $u_0 \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Furthermore, they obtained the uniqueness of the strong solutions when $2 \leqslant p < 2^* = \frac{2n}{n-2}$.

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a dissipative system is to analyze the existence and structure of its attractor. Generally speaking, the attractor has a very complicated geometry which reflects the complexity of the long-time behavior of the system. Therefore, it is necessary to study the existence of global attractors for the initial–boundary value problem for the complex Ginzburg–Landau equation. There have been many results for this equation in one- or two-dimensional space. For example, Lú [11] obtained the upper semi-continuity of approximations of attractors of the equation in one-dimensional space with $p = 4$. Temam [20] proved the existence of global attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ of the complex Ginzburg–Landau equation in the two-dimensional spaces. Ghidaglia and Héron [7], Doering et al. [5], Promislow [19] and Bu [2] studied the finite-dimensional attractor and related dynamical issues for (1.1)–(1.3) in the one- or two-dimensional spaces with nonlinearity $p = 2$ or $p = 6$. Nevertheless, for the case of $n \geqslant 3$, relevant results are scarce; the main reason is that some of the Sobolev interpolation inequalities used in one- or two-dimensional cases fail in the higher dimensional case and it is difficult to obtain the existence of absorbing sets. Furthermore, compared with the real Ginzburg–Landau equation, due to $(\lambda + i\alpha)\int_\Omega \nabla u \cdot \nabla(|u|^{p-2}\bar{u})\,dx$ and $(\kappa + i\beta)\int_\Omega \nabla\bar{u} \cdot \nabla(|u|^{p-2}u)\,dx$ is indefinite, it is difficult to obtain the existence of absorbing sets in $H_0^1(\Omega) \cap L^p(\Omega)$ without any restriction on $\frac{\alpha}{\lambda}$. Therefore, it is necessary to make some restriction on the ratio of the coefficients of the nonlinear term to overcome this difficulty.

The main purpose of this paper is to study the long time dynamical behavior for the complex Ginzburg–Landau equation (1.1)–(1.3) under the assumption

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in S\left(\frac{1}{c_p}, \frac{1}{c_p}\right), \tag{1.4}$$