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An estimate of the second moment of a sampling of the Riemann zeta function on the critical line

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ABSTRACT

We investigate the second moment of a random sampling $\zeta(1/2 + iX_t)$ of the Riemann zeta function on the critical line. Our main result states that if X_t is an increasing random sampling with gamma distribution, then for all sufficiently large t,

$$\mathbb{E}\left|\zeta(1/2 + iX_t)\right|^2 = \log t + O\left(\sqrt{\log t}\log\log t\right).$$

1. Introduction

This paper is concerned with the behaviour of the Riemann zeta function $\zeta(s)$ along the critical line $s = 1/2 + it$ by modelling the variable t with a random sampling. As is well known, the Riemann zeta function $\zeta(s)$ is defined as an analytic continuation of the function initially defined for all complex numbers $s = \sigma + it$ with real part greater than 1 by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Lifshits and Weber [3] studied the behaviour of the Riemann zeta function $\zeta(1/2 + it)$, when t is sampled by the Cauchy random walk. They used Cauchy distribution because the necessary moment expressions for Cauchy distribution are by far more explicit than in other cases. They remarked that they believe that the results similar to theirs are valid for sampling with a large class of random walks with discrete or continuous steps.

Here is our main result.

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Theorem 1. *Let X_t denote the gamma process with parameters $a = b = 1$. Then for all sufficiently large t,*

$$\mathbb{E}\left|\zeta(1/2 + iX_t)\right|^2 = \log t + O\left(\sqrt{\log t}\log\log t\right).$$

The gamma process has two positive parameters a and b. We set $a = b = 1$ for notational convenience. We note that the gamma process is increasing, its average value is t, and its variance is t. So, we use it to describe the situation how $\zeta(1/2 + it)$ behaves as t tends to infinity. In the next section, we recall the definition and basic properties of the gamma process. We will extensively use the Landau notation $f = O(g)$, which means that $|f(x)| \leqslant Cg(x)$ for some unspecified constant C. We also use the Vinogradov notation $f \ll g$; it is equivalent to $f = O(g)$.

We now give a few remarks.

Remark 1. Theorem 1 is a probabilistic analog of the famous result obtained by Hardy and Littlewood; as $T \to \infty$,

$$\frac{1}{T}\int_0^T \left|\zeta(1/2 + it)\right|^2 dt = \log T + O(1).$$

Remark 2. Jutila [2] obtained the following estimate for value distribution of the Riemann zeta function

$$\frac{1}{T}\operatorname{meas}\left(M_T(V)\right) \ll \exp\left(-\frac{\log^2 V}{\log\log T}\left(1 + O\left(\frac{\log V}{\log\log T}\right)\right)\right),$$

where $T \geqslant 2$, $1 \leqslant V \leqslant \log T$, and $M_T(V) = \{0 \leqslant t \leqslant T\colon\ |\zeta(1/2 + it)| \geqslant V\}$.

Remark 3. From Chebyshev's inequality, it is easy to see that the random sampling $\frac{|\zeta(1/2+iX_t)|}{\log t}$ converges to zero in probability. It is also a probabilistic analog of Jutila's result with $V = \log T$, although his result is much stronger.

Let us explain our method of proof. We begin by analytically extending the zeta function with suitable form, and then investigate the moment of the sampling $\zeta(\sigma + iX_t)$. Taking expectation is equivalent to considering the Fourier transform of probability measure of gamma process. The resulting equation is a type of oscillatory integrals. It is easy to estimate the first moment by applying repeated integration by parts. However, it is not so easy to estimate the second moment. The key idea in our strategy is divide and conquer algorithm based on recursion and iteration. We decompose sums and integrals into several pieces and then use a refined version of van der Corput's method to estimate each terms. The argument is technically elementary, but delicate.

2. Preliminaries

2.1. Gamma process

The gamma process $X_t^{a,b}$ plays a role as a natural continuous time analogue of independent and identically distributed sequence of positive increasing random variables. It is a pure-jump increasing Lévy process with independent gamma distributed increments with two positive parameters a and b. The law of $X_t^{a,b}$ is given by

$$d\mathcal{P}_{X_t^{a,b}}(x) = 1_{(0,\infty)}(x)\frac{b^{at}}{\Gamma(at)}x^{at-1}e^{-bx}\,dx.$$