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Reducibility of steady-state bifurcations in coupled cell systems

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ABSTRACT

A general theory for coupled cell systems was formulated recently by I. Stewart, M. Golubitsky and their collaborators. In their theory, a coupled cell system is a network of interacting dynamical systems whose coupling architecture is expressed by a directed graph called a coupled cell network. An equivalence relation on cells in a regular network (a coupled cell network with identical nodes and identical edges) determines a new network called quotient network by identifying cells in the same equivalence class and determines a quotient system as well. In this paper we develop an idea of reducibility of bifurcations in coupled cell systems associated with regular networks. A bifurcation of equilibria from subspace where states of all cells are equal is called a synchrony-breaking bifurcation. We say that a synchrony-breaking steady-state bifurcation is reducible in a coupled cell system if any bifurcation branch for the system is lifted from those for some quotient system. First, we give the complete classification of codimension-one synchrony-breaking steady-state bifurcations in 1-input regular networks (where each cell receives only one edge). Second, we show that under a mild condition on the multiplicity of critical eigenvalues, codimension-one synchrony-breaking steady-state bifurcations in generic coupled cell systems associated with an n-cell coupled cell network with D_n symmetry, a regular network, is reducible for $n > 2$.

1. Introduction

A general theory for coupled cell systems was introduced recently in I. Stewart et al. [6]. Since then the authors and their collaborators have been releasing many papers related to the theory. By their formulation a *coupled cell system* is a system of coupled ODEs whose coupling information is given by a *coupled cell network* that is essentially a directed graph whose nodes (or cells) represent states that evolve in time and whose edges (or couplings) represent interactions between those states. See [3,5,6] for more precise formulation.

In [1] the authors considered *synchrony-breaking bifurcations* in coupled cell systems, which is an analogue of the symmetry-breaking bifurcations in systems with symmetry. Such synchrony-breaking bifurcations are the main subject of this paper. We shall recall them briefly in the following paragraphs based on [1].

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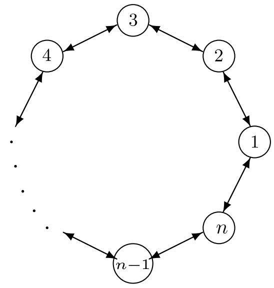

Fig. 1. n-Cell bidirectional ring (BR_n).

In this paper we study codimension-one synchrony-breaking bifurcations of steady-state solutions in coupled cell systems. We focus on a special class of coupled cell networks called *regular networks*, in which all cells are identical and couplings are also identical, in particular each cell has the same number of incoming edges called "inputs". For a regular network, define an associated ODE called an *admissible* vector field to the regular network as follows: Since the total number of cells is finite, we can enumerate the cells and let name the cells after its numbers. Let $x_j \in \mathbb{R}^k$ be the state variable of the j-th cell (or cell j), where k is the dimension of the internal dynamics in each cell, which is assumed to be identical. Then the j-th component of the admissible vector field has the form

$$\dot{x}_j = f(x_j, \overline{x_{\sigma_j(1)}, \ldots, x_{\sigma_j(v)}}), \quad j = 1, \ldots, n, \tag{1.1}$$

where the cell j receives inputs from the cells $\sigma_j(1), \ldots, \sigma_j(v)$. The $\sigma_j(i)$s are allowed to be equal to each other and even to j. The number v is called the *valency* of the network and it is constant for any choice of the cell j because each cell has the same number of inputs. The overbar indicates that the coupling coordinates are invariant under permutations of the coupling cells. This invariance is assumed, since we assume a unique type of coupling. Since there is only one type of node, we assume that the function $f : \mathbb{R}^k \times (\mathbb{R}^k)^v \to \mathbb{R}^k$ is independent of j.

Example *(n-Cell bidirectional ring).* Consider the n-cell regular network with valency 2, called a *bidirectional ring*, which is shown in Fig. 1.

The corresponding admissible vector field takes the form

$$\begin{cases} \dot{x}_1 = f(x_1, \overline{x_2, x_n}) \\ \dot{x}_2 = f(x_2, \overline{x_3, x_1}) \\ \quad\vdots \\ \dot{x}_n = f(x_n, \overline{x_1, x_{n-1}}) \end{cases} \tag{1.2}$$

where $f : \mathbb{R}^k \times (\mathbb{R}^k)^2 \to \mathbb{R}^k$ satisfies $f(a, b, c) = f(a, c, b)$.

We say that a coupled cell system exhibits *synchrony*, if two or more cells behave identically. A *polydiagonal* is a subspace Δ of the phase space $(\mathbb{R}^k)^n$ of coupled cell system which is defined by equalities among some cell coordinates. A *synchrony subspace* is a polydiagonal Δ that is flow-invariant for every admissible vector field associated with the coupled cell network. It is obvious that the subspace $\Delta_0 = \{(x, \ldots, x) \in (\mathbb{R}^k)^n\}$ given by setting all coordinates equal in a regular network yields a synchrony subspace. This Δ_0 is called the completely synchronous subspace.

We assume that an admissible vector field F has a completely synchronous equilibrium $X_0 \in \Delta_0$. Let $E^c = E^c_F(X_0)$ be the center subspace of $(dF)_{X_0}$. We say that the equilibrium X_0 has a *synchrony-breaking bifurcation*, if $E^c \setminus \Delta_0 \neq \emptyset$.