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Finite type domains with hyperbolic orbit accumulation points

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A B S T R A C T

In this paper, finite type domains with hyperbolic orbit accumulation points are studied. We prove, in case of $\mathbb{C}^2$, it has to be a (global) pseudoconvex domain, after an assumption of boundary regularity. Moreover, one of the applications will realize the classification of domains within this class, precisely the domain is biholomorphic to one of the ellipsoids $\{(z,w)\colon |z|^{2m}+|w|^2<1,\ m\in\mathbb{Z}^+\}$. This application generalizes [4] in which the boundary is assumed to be real analytic for the case of hyperbolic orbit accumulation points.

0. Introduction

Let Ω be a smooth bounded domain in $\mathbb{C}^n$ and $p\in\partial\Omega$. It was a long time since Greene–Krantz posted their conjecture in [7], which states that if p is a boundary orbit accumulation point, then p is a point of finite type. By orbit accumulation boundary point p, we mean a boundary point $p\in\partial\Omega$ such that $\lim_{j\to\infty} f_j(q)=p$ where $q\in\Omega$ and $f_j\in\operatorname{Aut}(\Omega)$. There are numerous works on this problem for 20 years by many mathematicians, e.g., we just mention some (in alphabet order), Eric Bedford, Jisoo Byun, Robert Greene, Kang-Tae Kim, Sung-Yeon Kim, Mario Landucci, Steven Krantz, Sergey Pinchuk, Jean-Christophe Yoccoz. Partial results have already been achieved, e.g. [2–4,7–13]. Among those, recently, Sung-Yeon Kim published the result in her paper [9] which proves the Greene–Krantz conjecture in case of hyperbolic orbit accumulation points. In this note, we consider the domain with noncompact automorphism groups from another point of view, namely, to check whether it is globally pseudoconvex. By pseudoconvex, we usually mean here weakly pseudoconvex, since a strongly pseudoconvex domain with noncompact automorphism groups will make the domain a ball by the well-known Wong–Rosay theorem (see [18] and [21]).

Let $\Omega\in\mathbb{C}^2$ be a domain with real analytic boundary. It was shown by Bedford–Pinchuk that noncompact automorphism group implies Ω is biholomorphic to one of the ellipsoids $\{(z,w)\colon |z|^{2m}+|w|^2,\ m\in\mathbb{Z}^+\}$. On can easily check that ellipsoids are globally pseudoconvex. However, if the problem passes to the category of smooth boundary, i.e. the defining function is C^∞, the answer is not so clear as for the domain with real analytic boundary. The difficulty is that some of the tools for real analytic boundary like Segre variety and analytic variety, cannot be used. Shortly after [2], Catlin pointed out (unpublished) that a pseudoconvex

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domain with boundary of finite type with noncompact automorphism group should be sufficient to be an ellipsoid (analytic is not necessary). However, one still wonders if "pseudoconvex" can be removed. The author will try to replace "pseudoconvex" with other assumptions, although he is unable to remove it completely so far.

In the present note, we mainly work on the following result.

Theorem 0.1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain with smooth boundary of finite type. Suppose that the Bergman kernel of Ω extends to $\overline{\Omega} \times \overline{\Omega}$ minus the boundary diagonal set as a locally bounded function. Let $p \in \partial\Omega$ be a hyperbolic orbit accumulation point. Then Ω is globally pseudoconvex.*

For the sake of completeness, we define the so-called "orbit accumulation points".

Definition 0.1. Let Ω be a smoothly bounded domain in $\mathbb{C}^2$. If there exist points $q \in \Omega$, $p \in \partial\Omega$ and a sequence $\{f_\nu\} \subset \mathrm{Aut}(\Omega)$ such that $f_\nu(q)$ converges to p. The point p is called an orbit accumulation point. If $f_\nu^{-1}(q)$ converges to another boundary point $\tilde{p} \in \partial\Omega - p$, where f_ν^{-1} is the inverse of f_ν, then p is called a hyperbolic orbit accumulation point.

The method of proof involves analysis of $\partial\Omega$ and the tools borrowed from CR geometry. We also try to write this note as concise as possible.

We should remark that in Theorem 0.1, "finite type" can be replaced with "boundary satisfying condition R in sense of Bell with p that is holomorphically simple (i.e. there is no complex variety through p that lies in the boundary)". Furthermore, the result is extended to higher dimensions in the author's forthcoming paper [14].

We also remark that for general case (not the hyperbolic accumulation pints), the method might not work. It is because the boundary might not be defined by a rigid equation then, even locally. Moreover, the condition of extension of the Bergman kernel to the boundary minus the diagonal set is verifiable when the $\bar{\partial}$-Neumann problem is pseudolocal.

1. Preliminaries

The Hilbert transform has a long history in both fields of one complex variable and several complex variables.

In particular, the Hilbert transform on the unit disc is most important. Let u be a real-valued function on $\partial\Delta$. Setting $z = re^{i\theta}$ with $0 \leqslant r < 1$ and $\zeta = e^{it}$, we define

$$T'u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_r(t)u\left(e^{i(\theta-t)}\right)\mathrm{d}t,$$

where

$$K_r(t) = \frac{2r\sin t}{1 - 2r\cos t + r^2}$$

is the Hilbert kernel (closely related to the well-known Poisson kernel).

Roughly speaking, the Hilbert transform is the limit function $T'u(re^{i\theta})$ as $r \to 1^-$. One can treat the following fact as the definition of the Hilbert transform.

For $u \in C^\alpha(\partial\Delta)$,

$$Tu\left(e^{i\theta}\right) = \mathrm{p.v.}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(e^{i(\theta-t)})}{\tan(t/2)}\mathrm{d}t.$$